

Note on the descriptions of the Euler-Poisson equations in various co-ordinate systems

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Abstract

In this note we derive the descriptions of the system of Euler-Poisson equations which governs the hydrodynamic evolution of gaseous stars in various co-ordinate systems. This note does not contain essentially new results for astrophysicists, but mathematically rigorous derivations cannot be found in the literatures written by physicists so that it will be useful to prepare details of rather stupidly honest derivations of the equations in various frames when we are going to push forward the mathematical research of the problem.

Key words and phrases. Euler-Poisson equations, gaseous star, axisymmetric solution, stellar oscillation, Lagrangian displacement.

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1 Euler-Poisson equations

1.1 Euler-Poisson equations and Newton potential

The Euler-Poisson equations which govern evolutions of a gaseous star are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad (1.1a)$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) + \nabla P = -\rho \nabla \Phi \quad (1.1b)$$

$$\Delta \Phi = 4\pi G \rho. \quad (1.1c)$$

The independent variable is $(t, \vec{x}) = (t, x^1, x^2, x^3) \in [0, T) \times \mathbb{R}^3$. G is a positive constant. The unknown functions are the density field $\rho = \rho(t, x)$, the pressure field $P = P(t, x)$, the gravitational potential $\Phi = \Phi(t, x)$, and the velocity field $\vec{v} = (v^1, v^2, v^3)(t, x)$.

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We are using the usual notations

$$\begin{aligned}\nabla \cdot (\rho \vec{v}) &= \sum_{k=1}^3 \frac{\partial}{\partial x^k} (\rho v^k), \\ (\vec{v} \cdot \nabla) v^j &= \sum_{k=1}^3 v^k \frac{\partial v^j}{\partial x^k}, \quad j = 1, 2, 3, \\ \nabla Q &= \left(\frac{\partial Q}{\partial x^1}, \frac{\partial Q}{\partial x^2}, \frac{\partial Q}{\partial x^3} \right) \quad \text{for } Q = P, \Phi \\ \Delta \Phi &= \sum_{k=1}^3 \frac{\partial^2 \Phi}{(\partial x^k)^2}.\end{aligned}$$

Also, we use the notation

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla = \frac{\partial}{\partial t} + \sum_{k=1}^3 v^k \frac{\partial}{\partial x^k}, \quad (1.2)$$

which rewrite (1.1a), (1.1b) as

$$\begin{aligned}\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} &= 0, \\ \rho \frac{D\vec{v}}{Dt} + \nabla P &= -\rho \nabla \Phi.\end{aligned}$$

We assume that P is a given smooth function of $\rho > 0$ such that $P > 0$, $dP/d\rho > 0$ for $\rho > 0$ and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}]_1) \quad (1.3)$$

as $\rho \rightarrow +0$, where A, γ are positive constants and $1 < \gamma \leq 2$. Here $[X]_1$ denotes a convergent power series of the form $\sum_{k \geq 1} a_k X^k$. Of course we consider $P = 0$ for $\rho = 0$.

Definition 1 *A solution $\rho = \rho(t, \vec{x}), \vec{v} = \vec{v}(t, \vec{x}), \Phi = \Phi(t, \vec{x})$ will be called a **compactly supported classical solution** if $\rho, \vec{v}, \Phi \in C^1([0, T) \times \mathbb{R}^3)$, $\Phi(t, \cdot) \in C^2(\mathbb{R}^3)$, $\rho \geq 0$ everywhere, and the support of $\rho(t, \cdot)$ is compact for $\forall t \in [0, T)$.*

Without loss of generality we assume $\vec{v}(t, \cdot)$ is bounded on \mathbb{R}^3 , since any modification of \vec{v} outside the support of ρ is free.

For any compactly supported classical solution the Laplace equation (1.1c) can be solved by the Newton potential

$$\Phi(t, \vec{x}) = -G \int \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d\mathcal{V}(\vec{x}'). \quad (1.4)$$

Here $d\mathcal{V}(\vec{x})$ denotes the usual volume element $dx^1 dx^2 dx^3$, and

$$|\vec{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

for $\vec{x} = (x^1, x^2, x^3)^T$.

In this note, we shall specify the solution Φ of (1.1c) by this Newton potential (1.4) for any compactly supported classical solution.

1.2 Conservation of mass, energy and angular momentum

It is well known that the total mass

$$M := \int \rho(t, \vec{x}) d\mathcal{V}(\vec{x}) \quad (1.5)$$

and the total energy

$$\begin{aligned} E &:= \int \left(\frac{1}{2} \rho |\vec{v}|^2 + \Psi(\rho) + \frac{1}{2} \rho \Phi \right) d\mathcal{V}(\vec{x}) \\ &= \int \left(\frac{1}{2} \rho |\vec{v}|^2 + \Psi(\rho) \right) d\mathcal{V}(\vec{x}) - \frac{G}{2} \int \int \frac{\rho(t, \vec{x}) \rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \end{aligned} \quad (1.6)$$

are constants with respect to t along any compactly supported classical solution.

Here the state quantities $u = u(\rho)$ (enthalpy) and $\Psi(\rho)$ is defined by

$$u = \int_0^\rho \frac{dP}{\rho}, \quad \Psi(\rho) = \int_0^\rho u d\rho. \quad (1.7)$$

Remark 1 *Note that*

$$u = \frac{A\gamma}{\gamma - 1} \rho^{\gamma-1}, \quad \Psi(\rho) = \frac{A\rho^\gamma}{\gamma - 1} = \frac{P}{\gamma - 1},$$

when $P = A\rho^\gamma$ exactly.

By way of precaution, let us verify the conservation of the total mass M and the total energy E .

First (1.1a) can be written as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v});$$

Hence

$$\begin{aligned} \frac{dM}{dt} &= - \int \nabla \cdot (\rho \vec{v}) d\mathcal{V} \\ &= - \int_{|\vec{x}|=R} (\rho \vec{v} | \vec{N}) d\mathcal{S} = 0, \end{aligned}$$

where the support of $\rho(t, \cdot)$ is supposed to be included in $\mathcal{D} = \{|\vec{x}| < R\}$ and $\vec{N}, d\mathcal{S}$ denote the outer normal vector and the surface area element of the boundary $\partial\mathcal{D} = \{|\vec{x}| = R\}$; This shows that $dM/dt = 0$.

Next, (1.1a)(1.1b) imply

$$\begin{aligned}
\frac{d}{dt} \int \frac{1}{2} \rho |\vec{v}|^2 d\mathcal{V} &= \frac{1}{2} \int \frac{\partial}{\partial t} \left(\rho \sum_k (v^k)^2 \right) d\mathcal{V} \\
&= \int \left(\rho \sum_k v^k \partial_t v^k + \frac{1}{2} \partial_t \rho \sum_k (v^k)^2 \right) d\mathcal{V} \\
&= - \int \left(\rho \sum_{j,k} v^k v^j \partial_j v^k + \sum_k v^k \partial_k P + \sum_k \rho v^k \partial_k \Phi + \right. \\
&\quad \left. + \frac{1}{2} \partial_j (\rho v^j) \sum_k (v^k)^2 \right) d\mathcal{V} \\
&= - \int \left(\frac{1}{2} \rho \sum_{j,k} v^j \partial_j (v^k)^2 + \sum_k v^k \partial_k P + \rho \sum_k v^k \partial_k \Phi + \right. \\
&\quad \left. + \frac{1}{2} \sum_j \partial_j (\rho v^j) \sum_k (v^k)^2 \right) d\mathcal{V} \\
&= - \int \frac{1}{2} \sum_{j,k} \partial_j \left[\rho v^j (v^k)^2 \right] - \int \sum_k (v^k \partial_k P + \rho v^k \partial_k \Phi) d\mathcal{V} \\
&= - \int \left(\sum_k \rho v^k \partial_k u + \sum_k \rho v^k \partial_k \Phi \right) d\mathcal{V} \\
&= - \int \left(\frac{\partial \Psi}{\partial t} + \sum_k \rho v^k \partial_k \Phi \right) d\mathcal{V} \\
&= - \frac{d}{dt} \int \Psi d\mathcal{V} + \int \sum_k \partial_k (\rho v^k) \Phi d\mathcal{V} \\
&= - \frac{d}{dt} \int \Psi d\mathcal{V} - \int \frac{\partial \rho}{\partial t} \Phi d\mathcal{V};
\end{aligned}$$

However

$$\begin{aligned}
\int \frac{\partial \rho}{\partial t} \Phi d\mathcal{V} &= -\mathsf{G} \int \int \partial_t \rho(t, \vec{x}) \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \\
&= -\frac{\mathsf{G}}{2} \frac{d}{dt} \int \frac{\rho(t, \vec{x}) \rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \\
&= \frac{1}{2} \frac{d}{dt} \int \rho \Phi d\mathcal{V};
\end{aligned}$$

This shows $dE/dt = 0$. Here we put $\vec{x} = (x^1, x^2, x^3)$ and $\vec{v} = (v^1, v^2, v^3)$, and denote $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x^j}$, while k, j run 1, 2, 3.

Moreover the angular momentum

$$\vec{J} := \int \vec{x} \times (\rho \vec{v}) d\mathcal{V}(\vec{x}) \quad (1.8)$$

is constant with respect to t along any compactly supported classical solution. Here

$$\vec{x} \times \vec{v} = \begin{bmatrix} x^2 v^3 - x^3 v^2 \\ x^3 v^1 - x^1 v^3 \\ x^1 v^2 - x^2 v^1 \end{bmatrix} \quad \text{for} \quad \vec{v} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}.$$

Let us show it. Note that (1.1b) can be written, under (1.1a), as

$$\frac{\partial}{\partial t}(\rho v^1) + \frac{\partial}{\partial x^1}(\rho (v^1)^2) + \frac{\partial}{\partial x^2}(\rho v^1 v^2) + \frac{\partial}{\partial x^3}(\rho v^1 v^3) + \frac{\partial P}{\partial x^1} = -\rho \frac{\partial \Phi}{\partial x^1} \quad (1.9a)$$

$$\frac{\partial}{\partial t}(\rho v^2) + \frac{\partial}{\partial x^1}(\rho v^2 v^1) + \frac{\partial}{\partial x^2}(\rho (v^2)^2) + \frac{\partial}{\partial x^3}(\rho v^2 v^3) + \frac{\partial P}{\partial x^2} = -\rho \frac{\partial \Phi}{\partial x^2} \quad (1.9b)$$

$$\frac{\partial}{\partial t}(\rho v^3) + \frac{\partial}{\partial x^1}(\rho v^3 v^1) + \frac{\partial}{\partial x^2}(\rho v^3 v^2) + \frac{\partial}{\partial x^3}(\rho (v^3)^2) + \frac{\partial P}{\partial x^3} = -\rho \frac{\partial \Phi}{\partial x^3} \quad (1.9c)$$

Then we have

$$\frac{dJ^1}{dt} = - \int (\vec{x} \times \nabla \Phi)^x \rho d\mathcal{V}(\vec{x}),$$

and so on, for $J = (J^1, J^2, J^3)^T$. Here

$$\vec{x} \times \nabla \Phi = \begin{bmatrix} (\vec{x} \times \nabla \Phi)^1 \\ (\vec{x} \times \nabla \Phi)^2 \\ (\vec{x} \times \nabla \Phi)^3 \end{bmatrix} = \begin{bmatrix} x^2 \frac{\partial \Phi}{\partial x^3} - x^3 \frac{\partial \Phi}{\partial x^2} \\ x^3 \frac{\partial \Phi}{\partial x^1} - x^1 \frac{\partial \Phi}{\partial x^3} \\ x^1 \frac{\partial \Phi}{\partial x^2} - x^2 \frac{\partial \Phi}{\partial x^1} \end{bmatrix}$$

On the other hand, the differentiation of the Newton potential (1.4) gives

$$\frac{\partial \Phi}{\partial x^3} = \mathsf{G} \int \frac{x^3 - (x^3)'}{|\vec{x} - \vec{x}'|^3} \rho(t, (x^1)', (x^2)', (x^3)') d\mathcal{V}(\vec{x}'),$$

and so on. Hence we see

$$\begin{aligned} \frac{dJ^1}{dt} &= \mathsf{G} \int \int \frac{-x^2(x^3 - (x^3)') + x^3(x^2 - (x^2)')}{|\vec{x} - \vec{x}'|^3} \rho(t, \vec{x}) \rho(t, \vec{x}') d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \\ &= \mathsf{G} \int \int \frac{x^2(x^3)'}{|\vec{x} - \vec{x}'|^3} \rho(t, \vec{x}) \rho(t, \vec{x}') d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \\ &\quad - \mathsf{G} \int \int \frac{x^3(x^2)'}{|\vec{x} - \vec{x}'|^3} \rho(t, \vec{x}) \rho(t, \vec{x}') d\mathcal{V}(\vec{x}) d\mathcal{V}(\vec{x}') \\ &= 0. \end{aligned}$$

By the same manner we can show $dJ^2/dt = dJ^3/dt = 0$. ■

2 Axisymmetric solutions

2.1 Co-ordinate system (ϖ, ϕ, z)

Let (ϖ, ϕ, z) be the cylindrical coordinates defined by

$$x^1 = \varpi \cos \phi, \quad x^2 = \varpi \sin \phi, \quad x^3 = z. \quad (2.1)$$

(This somewhat clumsy notation with ϖ is historically standard for this problem.)

Note that, while the polar co-ordinates are

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta,$$

we are taking $\varpi = r \sin \theta = \sqrt{r^2 - z^2}$.

We have

$$\begin{aligned} \frac{\partial}{\partial x^1} &= \frac{x^1}{\varpi} \frac{\partial}{\partial \varpi} - \frac{x^2}{\varpi^2} \frac{\partial}{\partial \phi}, \\ \frac{\partial}{\partial x^2} &= \frac{x^2}{\varpi} \frac{\partial}{\partial \varpi} + \frac{x^1}{\varpi^2} \frac{\partial}{\partial \phi}, \\ \frac{\partial}{\partial x^3} &= \frac{\partial}{\partial z}, \end{aligned}$$

since

$$\begin{aligned} \frac{\partial}{\partial \varpi} &= \frac{x^1}{\varpi} \frac{\partial}{\partial x^1} + \frac{x^2}{\varpi} \frac{\partial}{\partial x^2}, \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial x^3}, \\ \frac{\partial}{\partial \phi} &= -x^2 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^1}. \end{aligned}$$

Definition 2 A compactly supported solution ρ, \vec{v}, Φ will be said to be **axisymmetric** if $\partial \rho / \partial \phi = 0, \partial \Phi / \partial \phi = 0$, that is, $\rho = \rho(t, \varpi, z), \Phi = \Phi(t, \varpi, z)$ and if the velocity field \vec{v} is of the form

$$\vec{v} = \begin{bmatrix} \frac{V}{\varpi} x^1 - \Omega x^2 \\ \frac{V}{\varpi} x^2 + \Omega x^1 \\ W \end{bmatrix} \quad (2.2)$$

with $V = V(t, \varpi, z), W = W(t, \varpi, z), \Omega = \Omega(t, \varpi, z)$.

Note that if $\partial\rho/\partial\phi = 0$, then the Newton potential Φ given by (1.4) necessarily satisfies $\partial\Phi/\partial\phi = 0$.

Of course a spherically symmetric solution, for which

$$\rho = \rho(t, r), \quad \frac{V}{\varpi} = \frac{v(t, r)}{r}, \quad \Omega = 0, \quad W = \frac{v(t, r)}{r}z$$

with $r = \sqrt{\varpi^2 + z^2}$, is axisymmetric in this sense.

Let us derive the equations which govern axisymmetric solutions.

First we note that the following formula is easily verified:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial \varpi} + \Omega \frac{\partial}{\partial \phi} + W \frac{\partial}{\partial z}. \quad (2.3)$$

. In fact, we have

$$x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} = \varpi \frac{\partial}{\partial \varpi}, \quad -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} = \frac{\partial}{\partial \phi}.$$

Then the equation (1.1a) reads

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial V}{\partial \varpi} + \frac{V}{\varpi} + \frac{\partial W}{\partial z} \right) = 0. \quad (2.4)$$

Using the calculations

$$\begin{aligned} \frac{Dx^1}{Dt} &= \frac{V}{\varpi}x^1 - \Omega x^2, \\ \frac{Dx^2}{Dt} &= \frac{V}{\varpi}x^2 + \Omega x^1, \\ \frac{D\varpi}{Dt} &= V, \end{aligned}$$

we see that the equation (1.1b) reads

$$\rho \left[\frac{x^1}{\varpi} \frac{DV}{Dt} - x^2 \frac{D\Omega}{Dt} - 2 \frac{x^2}{\varpi} V \Omega - x^1 \Omega^2 \right] + \frac{\partial P}{\partial x^1} = -\rho \frac{\partial \Phi}{\partial x^1}, \quad (2.5a)$$

$$\rho \left[\frac{x^2}{\varpi} \frac{DV}{Dt} + x^1 \frac{D\Omega}{Dt} + 2 \frac{x^1}{\varpi} V \Omega - x^2 \Omega^2 \right] + \frac{\partial P}{\partial x^2} = -\rho \frac{\partial \Phi}{\partial x^2}, \quad (2.5b)$$

$$\rho \frac{DW}{Dt} + \frac{\partial P}{\partial x^3} = -\rho \frac{\partial \Phi}{\partial x^3}. \quad (2.5c)$$

Taking $\frac{1}{\varpi}(x^1 \cdot (2.5a) + x^2 \cdot (2.5b))$ and $-x^2 \cdot (2.5a) + x^1 \cdot (2.5b)$, we see that $(2.5a) \wedge (2.5b)$ is equivalent to

$$\rho \left[\frac{DV}{Dt} - \varpi \Omega^2 \right] + \frac{\partial P}{\partial \varpi} = -\rho \frac{\partial \Phi}{\partial \varpi}, \quad (2.6a)$$

$$\rho \frac{D}{Dt}(\varpi^2 \Omega) = 0. \quad (2.6b)$$

On the other hand the Laplace equation (1.1c) reads

$$\frac{1}{\varpi} \frac{\partial}{\partial \varpi} \left(\varpi \frac{\partial \Phi}{\partial \varpi} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho, \quad (2.7)$$

and the Newton potential (1.4) reads

$$\Phi(\varpi, z) = -G \int_{-\infty}^{+\infty} \int_0^\infty K_I(\varpi, \varpi', z - z') \rho(\varpi', z') \varpi' d\varpi' dz', \quad (2.8)$$

where

$$K_I(\varpi, \varpi', z - z') = 4 \int_0^{\pi/2} \frac{d\alpha}{\sqrt{(\varpi - \varpi')^2 + (z - z')^2 + 4\varpi\varpi' \sin^2 \alpha}} \quad (2.9)$$

Summing up, the full system which governs axisymmetric solutions is (2.4)(2.6a)(2.6b)(2.5c)(2.7), that is,

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial V}{\partial \varpi} + \frac{V}{\varpi} + \frac{\partial W}{\partial z} \right) = 0, \quad (2.10a)$$

$$\rho \left[\frac{DV}{Dt} - \varpi \Omega^2 \right] + \frac{\partial P}{\partial \varpi} = -\rho \frac{\partial \Phi}{\partial \varpi}, \quad (2.10b)$$

$$\rho \frac{DW}{Dt} + \frac{\partial P}{\partial z} = -\rho \frac{\partial \Phi}{\partial z}. \quad (2.10c)$$

$$\rho \frac{D}{Dt}(\varpi^2 \Omega) = 0. \quad (2.10d)$$

Here we note the operator D/Dt acting on functions which are axisymmetric reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial \varpi} + W \frac{\partial}{\partial z}.$$

Note that (2.10d) is a linear first order partial differential equation of Ω , provided that $\rho \neq 0$ and the components of the velocity fields V, W are known. Therefore, given V, W , the equation (2.10d) can be solved explicitly as follows.

Let

$$\Omega^0(\varpi, z) = \Omega|_{t=0} \quad (2.11)$$

be the initial data. For $t \in [0, T)$, $\varpi > 0$, $|z| < \infty$, we consider the solution $\tau \mapsto (\varphi(\tau ; t, \varpi, z), \psi(\tau ; t, \varpi, z))$ of the ordinary differential equations

$$\frac{d\varphi}{d\tau} = V(\tau, \varphi, \psi), \quad \frac{d\psi}{d\tau} = W(\tau, \varphi, \psi)$$

satisfying the initial conditions

$$\varphi(t ; t, \varpi, z) = \varpi, \quad \psi(t ; t, \varpi, z) = z.$$

Then the solution exists on the time interval $[0, t]$, provided that V, W are bounded, and Ω is given by

$$\Omega(t, \varpi, z) = \frac{\phi(0 ; t, \varpi, z)^2}{\varpi^2} \Omega^0(\varphi(0 ; t, \varpi, z), \psi(0 ; t, \varpi, z)). \quad (2.12)$$

Especially let us note that, if C is an arbitrary constant, then

$$\Omega(t, \varpi, z) = \frac{C}{\varpi^2} \quad (2.13)$$

satisfies the equation (2.10b), whatever ρ, V, W may be, when $\{ (2.10a), (2.10b), (2.10c), (2.7) \}$ turns out to be a closed system for only ρ, V, W, Φ . However, if $C \neq 0$, then $-\Omega x^2, \Omega x^1$ are unbounded at the axis $\varpi = 0$.

Let us calculate the angular momentum for the axisymmetric solution.

We see

$$\begin{aligned} \vec{x} \times \vec{v} &= \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \times \begin{bmatrix} \frac{V}{\varpi} x^1 - \Omega x^2 \\ \frac{V}{\varpi} x^2 + \Omega x^1 \\ W \end{bmatrix} = \\ &= \begin{bmatrix} -\frac{V}{\varpi} x^3 x^2 - \Omega x^3 x^1 + W x^2 \\ \frac{V}{\varpi} x^3 x^1 - \Omega x^3 x^2 - W x^1 \\ \Omega((x^1)^2 + (x^2)^2) \end{bmatrix} = \\ &= \begin{bmatrix} -V z \sin \phi - \Omega z \varpi \cos \phi + W \varpi \sin \phi \\ V z \cos \phi - \Omega z \varpi \sin \phi - W \varpi \cos \phi \\ \Omega \varpi^2 \end{bmatrix}. \end{aligned}$$

Integrating this, we get

$$\vec{J} = \begin{bmatrix} 0 \\ 0 \\ J \end{bmatrix}, \quad (2.14)$$

where

$$J = 2\pi \int_{-\infty}^{+\infty} \int_0^\infty \rho(t, \varpi, z) \Omega(t, \varpi, z) \varpi^3 d\varpi dz. \quad (2.15)$$

In fact, we note that

$$\int_{-\infty}^{+\infty} \int_0^{2\pi} \int_0^\infty \rho V z \sin \phi \cdot \varpi d\varpi d\phi dz = 0,$$

since $\int_0^{2\pi} \sin \phi d\phi = 0$ and so on.

We know that J is constant with respect to t .

Definition 3 Let $\rho = \rho(t, \varpi, z)$, $\vec{v} = (\frac{V}{\varpi}x^1 - \Omega x^2, \frac{V}{\varpi}x^2 + \Omega x^1, W)^T$ with $V = V(t, \varpi, z)$, $\Omega = \Omega(t, \varpi, z)$, $W = W(t, \varpi, z)$ be an axisymmetric solution. This solution is said to be **equatorially symmetric** if

$$\begin{aligned} \rho(t, \varpi, -z) &= \rho(t, \varpi, z), & V(t, \varpi, -z) &= V(t, \varpi, z), \\ \Omega(t, \varpi, -z) &= \Omega(t, \varpi, z), & W(t, \varpi, -z) &= -W(t, \varpi, z) \end{aligned}$$

for $\forall z$.

Then the potential Φ given by (1.4) necessarily satisfies

$$\Phi(t, \varpi, -z) = \Phi(t, \varpi, z)$$

for $\forall z$.

2.2 Co-ordinate system (r, ϕ, ζ)

Sometimes instead the co-ordinates (ϖ, z) one uses the co-ordinates (r, ζ) defined by

$$r = \sqrt{\varpi^2 + z^2}, \quad \zeta = \frac{z}{r} = \frac{z}{\sqrt{\varpi^2 + z^2}}. \quad (2.16)$$

Note that, while the polar co-ordinates are

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta,$$

we are taking $\zeta = \cos \theta$ so that

$$x^1 = r \sqrt{1 - \zeta^2} \cos \phi, \quad x^2 = r \sqrt{1 - \zeta^2} \sin \phi, \quad x^3 = r \zeta, \quad (2.17)$$

while

$$\frac{\partial}{\partial x^1} = \frac{x^1}{r} \frac{\partial}{\partial r} - \frac{x^1 \zeta}{r^2} \frac{\partial}{\partial \zeta} - \frac{x^2}{r^2(1 - \zeta^2)} \frac{\partial}{\partial \phi}, \quad (2.18a)$$

$$\frac{\partial}{\partial x^2} = \frac{x^2}{r} \frac{\partial}{\partial r} - \frac{x^2 \zeta}{r^2} \frac{\partial}{\partial \zeta} - \frac{x^1}{r^2(1 - \zeta^2)} \frac{\partial}{\partial \phi}, \quad (2.18b)$$

$$\frac{\partial}{\partial x^3} = \zeta \frac{\partial}{\partial r} + \frac{1 - \zeta^2}{r} \frac{\partial}{\partial \zeta}, \quad (2.18c)$$

since

$$\frac{\partial}{\partial r} = \frac{x^1}{r} \frac{\partial}{\partial x^1} + \frac{x^2}{r} \frac{\partial}{\partial x^2} + \zeta \frac{\partial}{\partial x^3}, \quad (2.19a)$$

$$\frac{\partial}{\partial \zeta} = -\frac{x^1 \zeta}{1 - \zeta^2} \frac{\partial}{\partial x^1} - \frac{x^2 \zeta}{1 - \zeta^2} \frac{\partial}{\partial x^2} + r \frac{\partial}{\partial x^3}, \quad (2.19b)$$

$$\frac{\partial}{\partial \phi} = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}. \quad (2.19c)$$

Let us introduce the velocity component variables v, w by

$$\vec{v} = \begin{bmatrix} \left(\frac{1}{r} v - \frac{\zeta}{1 - \zeta^2} w \right) x^1 - \Omega x^2 \\ \left(\frac{1}{r} v - \frac{\zeta}{1 - \zeta^2} w \right) x^2 + \Omega x^1 \\ \zeta v + r w \end{bmatrix}. \quad (2.20)$$

We are going to derive the equations,, assuming that v, w, Ω are functions of t, r, ζ , and independent of ϕ .

First we note that the formula

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial \zeta} + \Omega \frac{\partial}{\partial \phi}$$

holds. Note the following calculations:

$$\begin{aligned} \frac{Dx^1}{Dt} &= \frac{x^1}{r} v - \frac{x^1 \zeta}{1 - \zeta^2} w - x^2 \Omega, & \frac{Dx^2}{Dt} &= \frac{x^2}{r} v - \frac{x^2 \zeta}{1 - \zeta^2} w + x^1 \Omega, \\ \frac{Dr}{Dt} &= v, & \frac{D\zeta}{Dt} &= w, \\ \frac{D}{Dt} \left(\frac{x^1}{r} \right) &= -\frac{x^1 \zeta}{r(1 - \zeta^2)} w - \frac{x^2}{r} \Omega, \\ \frac{D}{Dt} \left(\frac{x^2}{r} \right) &= -\frac{x^2 \zeta}{r(1 - \zeta^2)} w + \frac{x^1}{r} \Omega, \\ \frac{D}{Dt} \left(\frac{x^1 \zeta}{1 - \zeta^2} \right) &= \frac{x^1 \zeta}{r(1 - \zeta^2)} v + \frac{x^1}{(1 - \zeta^2)^2} w - \frac{x^2 \zeta}{1 - \zeta^2} \Omega, \\ \frac{D}{Dt} \left(\frac{x^2 \zeta}{1 - \zeta^2} \right) &= \frac{x^2 \zeta}{r(1 - \zeta^2)} v + \frac{x^2}{(1 - \zeta^2)^2} w + \frac{x^1 \zeta}{1 - \zeta^2} \Omega. \end{aligned}$$

Using these calculations, we see that (1.1b) reads

$$\begin{aligned} \rho \left[\frac{x^1}{r} \frac{Dv}{Dt} - \frac{x^1 \zeta}{1 - \zeta^2} \frac{Dw}{Dt} - x^2 \frac{D\Omega}{Dt} \right. \\ \left. - \frac{2x^1 \zeta}{r(1 - \zeta^2)} vw - \frac{x^1}{(1 - \zeta^2)^2} w^2 - \frac{2x^2}{r} v\Omega + \frac{2x^2 \zeta}{1 - \zeta^2} w\Omega - x^1 \Omega^2 \right] + \frac{\partial P}{\partial x^1} + \rho \frac{\partial \Phi}{\partial x^1} = 0, \end{aligned} \quad (2.21a)$$

$$\begin{aligned} \rho \left[\frac{x^2}{r} \frac{Dv}{Dt} - \frac{x^2 \zeta}{1 - \zeta^2} \frac{Dw}{Dt} + x^1 \frac{D\Omega}{Dt} \right. \\ \left. - \frac{2x^2 \zeta}{r(1 - \zeta^2)} vw - \frac{x^2}{(1 - \zeta^2)^2} w^2 + \frac{2x^1}{r} v\Omega - \frac{2x^1 \zeta}{1 - \zeta^2} w\Omega - x^2 \Omega^2 \right] + \frac{\partial P}{\partial x^2} + \rho \frac{\partial \Phi}{\partial x^2} = 0, \end{aligned} \quad (2.21b)$$

$$\rho \left[\zeta \frac{Dv}{Dt} + r \frac{Dw}{Dt} + 2vw \right] + \frac{\partial P}{\partial x^3} + \rho \frac{\partial \Phi}{\partial x^3} = 0. \quad (2.21c)$$

Taking

$$\begin{bmatrix} \frac{x^1}{r} & \frac{x^2}{r} & \zeta \\ -\frac{x^1 \zeta}{r^2} & -\frac{x^2 \zeta}{r^2} & \frac{1 - \zeta^2}{r} \\ -\frac{x^2}{r^2(1 - \zeta^2)} & \frac{x^1}{r^2(1 - \zeta^2)} & 0 \end{bmatrix} \begin{bmatrix} (2.21a) \\ (2.21b) \\ (2.21c) \end{bmatrix},$$

we get the equivalent system:

$$\rho \left(\frac{Dv}{Dt} - \frac{r}{1 - \zeta^2} w^2 - r(1 - \zeta^2) \Omega^2 \right) + \frac{\partial P}{\partial r} + \rho \frac{\partial \Phi}{\partial r} = 0, \quad (2.22a)$$

$$\rho \left(\frac{Dw}{Dt} + \frac{2}{r} vw + \frac{\zeta}{1 - \zeta^2} w^2 + \zeta(1 - \zeta^2) \Omega^2 \right) + \frac{1 - \zeta^2}{r^2} \left(\frac{\partial P}{\partial \zeta} + \rho \frac{\partial \Phi}{\partial \zeta} \right) = 0, \quad (2.22b)$$

$$\rho \left(\frac{D\Omega}{Dt} + \frac{2}{r} v\Omega - \frac{2\zeta}{1 - \zeta^2} w\Omega \right) = 0. \quad (2.22c)$$

Summing up, the system of equations which governs the evolution of ρ, v, w, Ω

is:

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial v}{\partial r} + \frac{2}{r}v + \frac{\partial w}{\partial \zeta} \right) = 0, \quad (2.23a)$$

$$\rho \left(\frac{Dv}{Dt} - \frac{r}{1-\zeta^2}w^2 - r(1-\zeta^2)\Omega^2 \right) + \frac{\partial P}{\partial r} + \rho \frac{\partial \Phi}{\partial r} = 0, \quad (2.23b)$$

$$\rho \left(\frac{Dw}{Dt} + \frac{2}{r}vw + \frac{\zeta}{1-\zeta^2}w^2 + \zeta(1-\zeta^2)\Omega^2 \right) + \frac{1-\zeta^2}{r^2} \left(\frac{\partial P}{\partial \zeta} + \rho \frac{\partial \Phi}{\partial \zeta} \right) = 0, \quad (2.23c)$$

$$\rho \left(\frac{D\Omega}{Dt} + \frac{2}{r}v\Omega - \frac{2\zeta}{1-\zeta^2}w\Omega \right) = 0. \quad (2.23d)$$

Here we note that the operator D/Dt reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial \zeta},$$

when acting on axisymmetric functions ρ, v, w, Ω of t, r, ζ which do not depend on ϕ .

Of course, a solution $\rho = \rho(t, r, \zeta), v = v(t, r, \zeta), w = w(t, r, \zeta), \Omega = \Omega(t, r, \zeta), \Phi = \Phi(t, r, \zeta)$ is said to be **equatorially symmetric** if

$$\begin{aligned} \rho(t, r, -\zeta) &= \rho(t, r, \zeta), & v(t, r, -\zeta) &= v(t, r, \zeta), & w(t, r, -\zeta) &= -w(t, r, \zeta), \\ \Omega(t, r, -\zeta) &= \Omega(t, r, \zeta), & \Phi(t, r, -\zeta) &= \Phi(t, r, \zeta). \end{aligned}$$

for $\forall \zeta \in [-1, 1]$.

We note that (2.20) implies

$$\begin{aligned} |\vec{v}|^2 &= (v^1)^2 + (v^2)^2 + (v^3)^2 = \\ &= \left[\left(\frac{1}{r}v - \frac{\zeta}{1-\zeta^2}w \right) x^1 - \Omega x^2 \right]^2 + \left[\left(\frac{1}{r}v - \frac{\zeta}{1-\zeta^2}w \right) x^2 + \Omega x^1 \right]^2 + [\zeta v + rw]^2 = \\ &= v^2 + \frac{r^2}{1-\zeta^2}w^2 + r^2(1-\zeta^2)\Omega^2. \end{aligned} \quad (2.24)$$

Remark 2 Note that on the region $\rho > 0$ (2.23d) is equivalent to

$$\left[\frac{\partial}{\partial t} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial \zeta} \right] (r^2(1-\zeta^2)\Omega) = 0.$$

Therefore, along any solution $(r, \zeta) = (\varphi(t), \psi(t))$ of the equation

$$\frac{d\varphi}{dt} = v(t, \varphi(t), \psi(t)), \quad \frac{d\psi}{dt} = w(t, \varphi(t), \psi(t)),$$

we have

$$r^2(1 - \zeta^2)\Omega(t, r, \zeta) = \text{Const.}$$

In other words, if v, w and the initial data $\Omega^0(r, \zeta) = \Omega|_{t=0}$ are given, the solution Ω is given by

$$\Omega(t, r, \zeta) = \frac{\varphi(0)^2(1 - \psi(0)^2)}{r^2(1 - \zeta^2)}\Omega^0(\phi(0), \psi(0)),$$

where the couple of the functions $\tau \mapsto \varphi(\tau) = \varphi(\tau; t, r, \zeta), \tau \mapsto \psi(\tau) = \psi(\tau; t, r, \zeta)$ is the solution of

$$\begin{aligned} \frac{d\varphi}{d\tau} &= v(\tau, \varphi, \psi), & \frac{d\psi}{d\tau} &= w(\tau, \varphi, \psi), \\ \varphi(t; t, r, \zeta) &= r, & \psi(t; t, r, \zeta) &= \zeta. \end{aligned}$$

□

As for the Poisson equation and the Newton potential, we have

$$\Delta\Phi = \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial}{\partial \zeta} \right] \Phi = 4\pi G\rho \quad (2.25)$$

and

$$\Phi(t, r, \zeta) = -G \int_{-1}^1 \int_0^\infty K_{II}(r, \zeta, r', \zeta') \rho(t, r', \zeta') r'^2 dr' d\zeta', \quad (2.26)$$

where

$$K_{II}(r, \zeta, r', \zeta') = \int_0^{2\pi} \frac{d\beta}{\sqrt{r^2 + r'^2 - 2rr'(\sqrt{1 - \zeta^2}\sqrt{1 - \zeta'^2} \cos \beta + \zeta\zeta')}}. \quad (2.27)$$

Remark 3 When ρ is equatorially symmetric, that is, $\rho(r, -\zeta) = \rho(r, \zeta)$, then we have the formula

$$\begin{aligned} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d\mathcal{V}(\vec{x}') &= \int_{-1}^1 \int_0^\infty K_{II}(r, \zeta, r', \zeta') \rho(r', \zeta') r'^2 dr' d\zeta' = \\ &= 4\pi \int_0^1 \int_0^\infty \sum_{m=0}^\infty f_{2m}(r, r') P_{2m}(\zeta) P_{2m}(\zeta') \rho(r', \zeta') r'^2 dr' d\zeta', \end{aligned} \quad (2.28)$$

where

$$f_n(r, r') := \begin{cases} \frac{1}{r} \left(\frac{r'}{r} \right)^n & \text{for } r' < r \\ \frac{1}{r'} \left(\frac{r}{r'} \right)^n & \text{for } r < r' \end{cases} \quad (2.29)$$

and P_n are the Legendre polynomials.

3 Stationary solutions

3.1 Axisymmetric stationary solutions

Let us use the co-ordinates (t, ϖ, z) .

Definition 4 *An axisymmetric solution is said to be **stationary** if $\rho = \rho(\varpi, z)$, $V = W = 0$, $\Omega = \Omega(\varpi, z)$, $\Phi = \Phi(\varpi, z)$.*

The equations are reduced to

$$-\rho\Omega^2 + \frac{1}{\varpi} \frac{\partial P}{\partial \varpi} = -\frac{\rho}{\varpi} \frac{\partial \Phi}{\partial \varpi}, \quad (3.1a)$$

$$\frac{\partial P}{\partial z} = -\rho \frac{\partial \Phi}{\partial z}, \quad (3.1b)$$

$$\Delta \Phi = 4\pi G \rho, \quad (3.1c)$$

where

$$\Delta \Phi = \frac{1}{\varpi} \frac{\partial}{\partial \varpi} \left(\varpi \frac{\partial \Phi}{\partial \varpi} \right) + \frac{\partial^2 \Phi}{\partial z^2}. \quad (3.2)$$

In our barotropic case, that is, when $P = P(\rho)$ is a given function of ρ , we have $\partial \Omega / \partial z = 0$, that is, $\Omega = \Omega(\varpi)$.

In fact

$$\frac{\partial}{\partial z} \left[\frac{\varpi}{\rho} \times (3.1a) \right] - \frac{\partial}{\partial \varpi} \left[\frac{1}{\rho} \times (3.1b) \right]$$

implies $\partial(\varpi\Omega^2)/\partial z = 0$, keeping in mind that $du = dP/\rho$ so that

$$\frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial P}{\partial \varpi} \right) = \frac{\partial}{\partial \varpi} \left(\frac{1}{\rho} \frac{\partial P}{\partial z} \right),$$

provided that $u \in C^2$.

If Ω is a nonzero constant, then the stationary solution is called a **solid rotation**, and if Ω is not a constant, then the solution is called a **differential rotation**.

If we use the variable

$$u = \int_0^\rho \frac{dP}{\rho},$$

then on the region where $\rho > 0$, the system (3.1a)(3.1b)(3.1c) is reduced to

$$-\varpi\Omega^2 + \frac{\partial u}{\partial \varpi} = -\frac{\partial \Phi}{\partial \varpi}, \quad (3.3a)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial \Phi}{\partial z}, \quad (3.3b)$$

$$\Delta \Phi = 4\pi G \rho. \quad (3.3c)$$

Then we have

$$-\Delta u = 4\pi G\rho + f(\varpi), \quad (3.4)$$

where

$$f(\varpi) := -\frac{1}{\varpi} \frac{d}{d\varpi} (\varpi^2 \Omega^2). \quad (3.5)$$

since $\partial\Omega/\partial z = 0$ identically, that is, since $\Omega = \Omega(\varpi)$.

And (3.3a)(3.3b) imply

$$-\int_0^\varpi \Omega(\varpi')^2 \varpi' d\varpi' + u + \Phi = \text{Constant}. \quad (3.6)$$

Note that, if Ω is a constant, then $f = -2\Omega^2$, and we have

$$-\Delta u = 4\pi G\rho - 2\Omega^2, \quad (3.7)$$

and

$$u + \Phi = \frac{\varpi^2}{2} \Omega^2 + \text{Const}. \quad (3.8)$$

On the other hand, if we use the co-ordinates (t, r, ζ) defined by

$$r = \sqrt{\varpi^2 + z^2}, \quad \zeta = \frac{z}{r} = \frac{z}{\sqrt{\varpi^2 + z^2}}, \quad (3.9)$$

the equations turn out to be

$$-\rho(1 - \zeta^2)r\Omega^2 + \frac{\partial P}{\partial r} = -\rho \frac{\partial \Phi}{\partial r} \quad (3.10a)$$

$$\rho\zeta r^2\Omega^2 + \frac{\partial P}{\partial \zeta} = -\rho \frac{\partial \Phi}{\partial \zeta} \quad (3.10b)$$

$$\Delta \Phi = 4\pi G\rho, \quad (3.10c)$$

where

$$\Delta \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial \Phi}{\partial \zeta}.$$

If we use the variable

$$u = \int_0^\rho \frac{dP}{\rho},$$

then on the region $\rho > 0$ the system (3.10a)(3.10b)(3.10c) turns out to be

$$-(1 - \zeta^2)r\Omega^2 + \frac{\partial u}{\partial r} = -\frac{\partial \Phi}{\partial r}, \quad (3.11a)$$

$$\zeta r^2\Omega^2 + \frac{\partial u}{\partial \zeta} = -\frac{\partial \Phi}{\partial \zeta}. \quad (3.11b)$$

$$\Delta \Phi = 4\pi G\rho. \quad (3.11c)$$

Suppose that Ω is a constant. Then we have

$$-\Delta u = 4\pi G\rho - 2\Omega^2, \quad (3.12)$$

and

$$u + \Phi = \frac{1}{2}r^2(1 - \zeta^2)\Omega^2 + \text{Const.} \quad (3.13)$$

Clearly we see that $(3.11a) \wedge (3.11b) \wedge (3.11c) \Leftrightarrow (3.11c) \wedge (3.13)$.

3.2 Normalization

Suppose $P = A\rho^\gamma$ exactly and $u = \frac{A\gamma}{\gamma-1}\rho^{\gamma-1}$.

Let us normalize the variables. Taking $r = \alpha\bar{r}$, $u = \beta\bar{u}$, $\Phi = \beta\bar{\Phi}$, we have on the region $u > 0$

$$-\frac{\varepsilon}{2}(1 - \zeta^2)\bar{r} + \frac{\partial\bar{u}}{\partial\bar{r}} = -\frac{\partial\bar{\Phi}}{\partial\bar{r}}, \quad (3.14a)$$

$$\frac{\varepsilon}{2}\zeta\bar{r}^2 + \frac{\partial\bar{u}}{\partial\zeta} = -\frac{\partial\bar{\Phi}}{\partial\zeta}, \quad (3.14b)$$

$$-\bar{\Delta}\bar{u} = \bar{u}^\nu - \varepsilon, \quad (3.14c)$$

with

$$\nu = \frac{1}{\gamma-1}, \quad 4\pi G\left(\frac{\gamma-1}{A\gamma}\right)^\nu \alpha^2 \beta^{\frac{2-\gamma}{\gamma-1}} = 1, \quad 2\Omega^2 \alpha^2 / \beta = \varepsilon. \quad (3.15)$$

In order to fix the idea, we may put

$$\beta = u_c = \frac{A\gamma}{\gamma-1}\rho_c^{\gamma-1},$$

ρ_c being the central density, when $\bar{u}(\vec{0}) = 1$. Then

$$\alpha = \sqrt{\frac{A\gamma}{4\pi G(\gamma-1)}}\rho_c^{\frac{\gamma-2}{2}}, \quad \varepsilon = \frac{\Omega^2}{2\pi G\rho_c}, \quad \rho(r, \zeta) = \rho_c \bar{u}(r/\alpha, \varepsilon)^\nu,$$

and the Newton potential is

$$\bar{\Phi}(\bar{r}, \zeta) = -\frac{1}{4\pi} \int_{-1}^1 \int_0^\infty K_{II}(\bar{r}, \bar{r}', \zeta, \zeta') \bar{u}_\#(\bar{r}', \zeta')^\nu \bar{r}'^2 d\bar{r}' d\zeta', \quad (3.16)$$

where $\bar{u}_\# = \max\{\bar{u}, 0\}$, since

$$\bar{\Delta}\bar{\Phi} = \bar{u}_\#^\nu. \quad (3.17)$$

Of course we assume that $\bar{u}(0, 0) = \bar{u}(0, \zeta) \forall \zeta \in [-1, 1]$, since it is nothing but $\bar{u}(\vec{0})$. On the other hand, it easy to see the potential $\bar{\Phi}(\bar{r}, \zeta)$ defined by (3.16) satisfies $\bar{\Phi}(0, 0) = \bar{\Phi}(0, \zeta) \forall \zeta \in [-1, 1]$. In fact we have

$$\frac{\partial}{\partial\zeta} K_{II}(0, r', \zeta, \zeta') = 0 \quad \text{for } \forall(r', \zeta, \zeta').$$

Now (3.14a)^(3.14b) is equivalent to

$$\bar{\Phi} + \bar{u} = \frac{\varepsilon}{4}(1 - \zeta^2)\bar{r}^2 + \text{Const.} \quad (3.18)$$

which should hold on the region $\bar{u} > 0$.

Remark 4 *Let us denote \bar{u}, \bar{r} and so on by u, r and so on. The equation (3.14c) :*

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \zeta}(1 - \zeta^2)\frac{\partial}{\partial \zeta}\right)u = u^\nu - \varepsilon,$$

admits a spherically symmetric solution (near to the Lane-Emden function) $u = U(r)$ such that $U(r) > 0 \Leftrightarrow 0 \leq r < R(< +\infty)$, provided that $\gamma > 6/5$ and $\Omega^2 \ll 1$, that is, $\varepsilon \ll 1$, which solves

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{dU}{dr} = U^\nu - \varepsilon, \quad U = 1 + O(r^2) \quad \text{as } r \rightarrow 0.$$

But this does not give a solution to our problem if $\Omega \neq 0$, that is, $\varepsilon \neq 0$. In fact, the Newton potential

$$\Phi(r, \zeta) = -\frac{1}{4\pi} \int_{-1}^1 \int_0^R K_{II}(r, r', \zeta, \zeta') U(r')^\nu r'^2 dr' d\zeta'$$

associated to this spherically symmetric U is spherically symmetric, that is, $\partial\Phi/\partial\zeta = 0$. Then $\partial U/\partial\zeta = \partial\Phi/\partial\zeta = 0$ would imply $\varepsilon = 0$ in view of (3.14b), a contradiction.

3.3 Spherically symmetric stationary solutions

Here let us recall the well known results on spherically symmetric stationary solutions $\rho = \rho(r), V = W = \Omega = 0, \Phi = \Phi(r)$, when the equation of state is the exact γ -law : $P = A\rho^\gamma$. See [3] and [5].

The equations are

$$\begin{aligned} \frac{dP}{dr} &= -\rho \frac{d\Phi}{dr}, \\ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\Phi}{dr} &= 4\pi G \rho \end{aligned}$$

and the Newton potential is

$$\begin{aligned} \Phi(r) &= -G \int_0^\infty K_{III}(r, r') \rho(r') r'^2 dr', \\ K_{III}(r, r') &= 4\pi \cdot \min\left(\frac{1}{r}, \frac{1}{r'}\right) \end{aligned}$$

Note that

$$\Phi(r) = -\frac{GM}{r} \quad \text{for } r \geq R$$

if $\rho(r) = 0$ for $r \geq R$ and $M = 4\pi \int_0^R \rho(r)r^2 dr$.

The **Lane-Emden equation of index ν** is

$$-\frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \frac{d\theta}{d\xi} = \theta^\nu$$

and the **Lane-Emden function of index ν** is the solution of this equation which satisfies

$$\theta = 1, \quad \frac{d\theta}{d\xi} = 0 \quad \text{at } \xi = 0,$$

which will be denoted by $\theta(\xi ; \nu)$. If $1 \leq \nu < 5$, there is a finite $\xi_1(\nu)$ such that $\theta(\xi ; \nu) > 0$ for $0 \leq \xi < \xi_1(\nu)$ and $\theta(\xi_1(\nu) ; \nu) = 0$. The numerical table on [3, Dover Ed., p.96] reads

ν	$\xi_1(\nu)$	$\mu_1(\nu)$
1.0	3.14159	3.14159
1.5	3.65375	2.71406
2.0	4.35287	2.41105
2.5	5.35528	2.18720
3.0	6.89685	2.01824
3.5	9.53581	1.89056
4.0	14.97155	1.79723
4.5	31.83646	1.73780
5.0	∞	1.73205

Here

$$\mu_1(\nu) := \int_0^{\xi_1(\nu)} \theta(\xi ; \nu)^\nu \xi^2 d\xi = -\xi^2 \frac{d\theta(\xi ; \nu)}{d\xi} \Big|_{\xi=\xi_1(\nu)}.$$

The equilibrium with the central density $\rho|_{r=0} = \rho_c$ is given by the formula

$$\rho = \rho_c \theta(r/\alpha ; \nu)^\nu, \quad u = \frac{A\gamma}{\gamma-1} \rho_c^{\gamma-1} \theta(r/\alpha ; \nu)$$

with

$$\nu = \frac{1}{\gamma-1}, \quad \alpha = \sqrt{\frac{A\gamma}{4\pi G(\gamma-1)}} \rho_c^{-\frac{2-\gamma}{2}}.$$

The total mass is

$$M = 4\pi \left(\frac{A\gamma}{4\pi G(\gamma-1)} \right)^{3/2} \rho_c^{\frac{3\gamma-4}{2}} \mu_1(\nu)$$

and the radius is

$$R = \sqrt{\frac{A\gamma}{4\pi G(\gamma-1)}} \rho_c^{-\frac{2-\gamma}{2}} \xi_1(\nu).$$

Let $1 \leq \nu < 5 (\Leftrightarrow 6/5 < \gamma \leq 2)$.

If $\nu \neq 3 (\Leftrightarrow \gamma \neq 4/3)$, the equilibrium is uniquely determined for each given total mass M . However if $\nu = 3 (\Leftrightarrow \gamma = 4/3)$, the total mass does not depend on the central density, so infinitely many equilibria correspond to the same critical mass. This situation is same for the total energy. Actually we have

$$\begin{aligned} E &= \frac{4-3\gamma}{\gamma-1} \int P \cdot 4\pi r^2 dr \\ &= \frac{4\gamma-3}{\gamma-1} 4\pi A \left(\frac{A\gamma}{4\pi G(\gamma-1)} \right)^{3/2} \frac{\rho_c^{\frac{5\gamma-6}{2}}}{\rho_c^{\frac{5\gamma-6}{2}}} \int_0^{\xi_1(\nu)} \theta(\xi; \nu)^{\nu+1} \xi^2 d\xi \end{aligned}$$

vanishes for any central density if $\nu = 3 (\Leftrightarrow \gamma = 4/3)$.

In fact the identity

$$\begin{aligned} E &= \int \frac{P}{\gamma-1} \cdot 4\pi r^2 dr + \frac{1}{2} \int \rho \Phi \cdot 4\pi r^2 dr \\ &= \frac{4-3\gamma}{\gamma-1} \int P \cdot 4\pi r^2 dr \end{aligned}$$

can be shown as follows:

$$\begin{aligned} \text{Let } m &= \int_0^r \rho \cdot 4\pi r^2 dr. \text{ Then we see} \\ \frac{1}{2} \int \rho \Phi \cdot 4\pi r^2 dr &= \frac{1}{2} \int_0^R \Phi dm = \frac{1}{2} \Phi(R)M - \frac{1}{2} \int_0^R m \frac{d\Phi}{dr} dr \\ &= -\frac{1}{2} \frac{GM}{R} - \frac{1}{2} \int_0^R \frac{Gm^2}{r^2} dr \\ &\quad \left(\text{since } r^2 \frac{d\Phi}{dr} = 4\pi G \int_0^r \rho r^2 dr = Gm \right) \\ &= -\frac{1}{2} \int_0^R \frac{2Gm}{r} \frac{dm}{dr} dr = -G \int_0^R 4\pi \rho m r dr = \int_0^R \frac{dP}{dr} 4\pi r^3 dr \\ &\quad \left(\text{since } \frac{dP}{dr} = -\rho \frac{d\Phi}{dr} = -\frac{G\rho m}{r^2} \right) \\ &= -3 \int_0^R P \cdot 4\pi r^2 dr. \end{aligned}$$

Thus we have the required identity. ■

4 Lagrangian variation and rotating frame

In this section we follow the Lagrangian variation of the equations which has been adopted by astrophysicists.

4.1 Lagrangian change operator and the convective change operator

Let (t, x^1, x^2, x^3) be a co-ordinate system of the space-time, which is not necessarily an inertial Galilean system.

N. R. Lebovitz [6] introduced the operator Δ , which we shall call the ‘**Lebovitz’ Lagrangian change operator**’, defined by

$$\Delta Q(t, x) = Q(t, x + \xi(t, x)) - \bar{Q}(t, x), \quad (4.1)$$

where Q and \bar{Q} are the values of a physical quantity in the perturbed and unperturbed flows. The quantity ξ , which we shall call the ‘**Lebovitz’ displacement**’, is defined as follows: Let $t \mapsto x = \varphi(t; s, y) = (\varphi^j(t; s, y) | j = 1, 2, 3)$ and $t \mapsto x = \bar{\varphi}(t; s, y) = (\bar{\varphi}^j(t; s, y) | j = 1, 2, 3)$ be the perturbed and unperturbed flow of the fluid particle passing the point $(t, x) = (s, y)$, that is,

$$\frac{\partial}{\partial t} \varphi^j(t; s, y) = u^j(t, \varphi(t; s, y)), \quad (4.2a)$$

$$\varphi^j(s; s, y) = y^j, \quad (4.2b)$$

and

$$\frac{\partial}{\partial t} \bar{\varphi}^j(t; s, y) = \bar{u}^j(t, \bar{\varphi}(t; s, y)), \quad (4.3a)$$

$$\bar{\varphi}^j(s; s, y) = y^j, \quad (4.3b)$$

where $u(t, x) = (u^j(t, x) | j = 1, 2, 3)$ and $\bar{u}(t, x) = (\bar{u}^j(t, x) | j = 1, 2, 3)$ are the perturbed and unperturbed velocity fields; Then N. R. Lebovitz defined the displacement $\xi(t, x) = (\xi^1(t, x), \xi^2(t, x), \xi^3(t, x))$ by

$$\xi^j(t, x) := \varphi^j(t; t_0, \bar{\varphi}(t_0; t, x)) - x^j, \quad j = 1, 2, 3, \quad (4.4)$$

where t_0 is an arbitrarily fixed time instant.

By contrast the ‘**Eulerian difference operator**’ δ is defined by [7, p. 293, (2)] as

$$\delta Q(t, x) = Q(t, x) - \bar{Q}(t, x).$$

Clearly when the derivatives of Q are Lipschitz continuous we have

$$\Delta Q = \delta Q + (\xi | \nabla) Q + O(|\xi|^2),$$

where

$$(\xi | \nabla) Q(t, x) = \sum_k \xi^k(t, x) \frac{\partial}{\partial x^k} Q(t, x).$$

[7, (3)].

Note that in some astrophysical literatures, the Lagrangian change operator Δ is defined as $\delta + (\xi | \nabla)$. We should be carefull to avoid confusions.

On the other hand the so-called ‘**convective rate of change operator**’ D/Dt is defined as usually as follows:

$$\frac{D}{Dt}Q(t, x) = \frac{\partial}{\partial t}Q(t, x) + \sum_j u^j(t, x) \frac{\partial}{\partial x^j}Q(t, x), \quad (4.5)$$

when the perturbed velocity field $u = (u^j | j = 1, 2, 3)$ is given. In parallel with this D/Dt , we write

$$\overline{\frac{D}{Dt}} = \frac{\partial}{\partial t} + \sum_j \bar{u}^j(t, x) \frac{\partial}{\partial x^j}.$$

Of course

$$\frac{\partial}{\partial t}Q(t, \varphi(t; s, y)) = \frac{DQ}{Dt}(t, \varphi(t; s, y)) \quad (4.6)$$

along the flow $x = \varphi(t; s, y)$ for any fixed s, y .

Now, as for the commutation of the operators Δ and D/Dt , D. Lynden-Bell and J. P. Ostriker claimed the following formula.

Formula 1 *For any quantity Q the identity*

$$\Delta \frac{D}{Dt}Q = \overline{\frac{D}{Dt}}\Delta Q. \quad (4.7)$$

holds.

See [7, p.294, (9)]. But their proof described in [7] seems to fall back on a kind of too much literary rhetoric. So, here let us describe a simple and stupidly honest proof of this identity.

We are going to prove that

$$\clubsuit := \overline{\frac{D}{Dt}}\Delta Q(t, x) - \Delta \frac{D}{Dt}Q(t, x) \quad (4.8)$$

vanishes everywhere.

By the definition we have

$$\begin{aligned} \Delta \frac{D}{Dt}Q(t, x) &= \frac{DQ}{Dt}(t, x + \xi) - \overline{\frac{D}{Dt}}\bar{Q}(t, x) \\ &= \partial_t Q(t, x + \xi) + \sum_j u^j(t, x + \xi) \partial_{z^j} Q(t, z)|_{z=x+\xi} \\ &\quad - \partial_t \bar{Q}(t, x) - \sum_j \bar{u}^j(t, x) \partial_{x^j} \bar{Q}(t, x). \end{aligned} \quad (4.9)$$

Let us calculate $\overline{D/Dt}\Delta Q$. We have

$$\begin{aligned}
\overline{\frac{D}{Dt}}\Delta Q(t, x) &= \frac{\partial}{\partial t}[\Delta Q(t, x)] + \sum_j \bar{u}^j(t, x) \frac{\partial}{\partial x^j}[\Delta Q(t, x)] \\
&= \partial_t Q(t, x + \xi) + \sum_j \partial_t \xi^j(t, x) \partial_{z^j} Q(t, z)|_{z=x+\xi} - \partial_t \bar{Q}(t, x) \\
&\quad + \sum_j \bar{u}^j(t, x) \frac{\partial}{\partial x^j}[Q(t, x + \xi)] - \sum_j \bar{u}^j(t, x) \partial_{x^j} \bar{Q}(t, x) \\
&= \partial_t Q(t, x + \xi) + \sum_j \partial_t \xi^j(t, x) \partial_{z^j} Q(t, z)|_{z=x+\xi} - \partial_t \bar{Q}(t, x) \\
&\quad + \sum_j \bar{u}^j(t, x) (\delta_j^k + \partial_{x^j} \xi^k(t, x)) \partial_{z^j} Q(t, z)|_{z=x+\xi} - \sum_j \bar{u}^j(t, x) \partial_{x^j} \bar{Q}(t, x).
\end{aligned}$$

Therefore it holds that

$$\begin{aligned}
\clubsuit &= \sum_j \partial_t \xi^j(t, x) \partial_{z^j} Q(t, z)|_{z=x+\xi} + \sum_j \bar{u}^j(t, x) \partial_{z^j} Q(t, z)|_{z=x+\xi} \\
&\quad + \sum_{j,k} \bar{u}^j(t, x) \partial_{x^j} \xi^k(t, x) \partial_{z^k} Q(t, z)|_{z=x+\xi} \\
&\quad - \sum_j u^j(t, x + \xi) \partial_{z^j} Q(t, z)|_{z=x+\xi}.
\end{aligned} \tag{4.10}$$

On the other hand, differentiating

$$\xi^k(t, x) = \varphi^k(t; t_0, \bar{\varphi}(t_0; t, x)) - x^k \tag{4.11}$$

with respect to t , we have

$$\partial_t \xi^k(t, x) = u^k(t, x + \xi) + J_0^k(t, x), \tag{4.12}$$

where

$$J_0^k(t, x) := \sum_\ell \frac{\partial}{\partial t}[\bar{\varphi}^\ell(t_0; t, x)] \frac{\partial}{\partial y^\ell}[\varphi^k(t; t_0, y)]|_{y=\bar{\varphi}(t_0; t, x)}. \tag{4.13}$$

By differentiating (4.11) with respect to x^j , we have

$$\partial_{x^j} \xi^k(t, x) = J_j^k(t, x) - \delta_j^k, \tag{4.14}$$

where

$$J_j^k(t, x) = \sum_\ell \frac{\partial}{\partial x^j}[\bar{\varphi}^\ell(t_0; t, x)] \frac{\partial}{\partial y^\ell}[\varphi^k(t; t_0, y)]|_{y=\bar{\varphi}(t_0; t, x)}. \tag{4.15}$$

Substituting (4.12) (4.14) into (4.10), we see

$$\clubsuit = \sum_k \left(J_0^k(t, x) + \sum_j \bar{u}^j(t, x) J_j^k(t, x) \right) \partial_{z^k} Q(t, z)|_{z=x+\xi}, \tag{4.16}$$

where J_0^k and J_j^k are given by (4.13) and (4.15). Let us put

$$C^\ell(t, x) := \frac{\partial}{\partial t}[\bar{\varphi}^\ell(t_0; t, x)] + \sum_j \bar{u}^j(t, x) \frac{\partial}{\partial x^j}[\bar{\varphi}^\ell(t_0; t, x)], \quad (4.17)$$

which gives

$$J_0^k(t, x) + \sum_j \bar{u}^j(t, x) J_j^k(t, x) = \sum_\ell C^\ell(t, x) \frac{\partial}{\partial y^\ell}[\varphi^k(t; t_0, y)]|_{y=\bar{\varphi}(t_0; t, x)}. \quad (4.18)$$

Thus in order to prove that \clubsuit vanishes, it is sufficient to prove that $C^\ell(t, x) = 0$ for $\forall \ell = 1, 2, 3$. So, we are going to evaluate terms of $C^\ell(t, x)$.

Let us recall (4.3a)(4.3b) and consider

$$\bar{\psi}_0^j(\tau, s, y) := \frac{\partial}{\partial s} \bar{\varphi}^j(\tau; s, y), \quad (4.19)$$

which can evaluate

$$\frac{\partial}{\partial t}[\bar{\varphi}^\ell(t_0; t, x)] = \bar{\psi}_0^\ell(\tau, s, y)|_{(\tau, s, y)=(t_0, t, x)}. \quad (4.20)$$

The functions $\bar{\psi}_0^j$ is determined by the initial value problem

$$\frac{\partial}{\partial \tau} \bar{\psi}_0^j(\tau, s, y) = \sum_\ell \bar{U}_\ell^j(\tau, \varphi(\tau; s, y)) \bar{\psi}_0^\ell(\tau, s, y) \quad (4.21a)$$

$$\bar{\psi}_0^j(s, s, y) = -\bar{u}^j(s, y), \quad (4.21b)$$

where

$$\bar{U}_\ell^j(t, x) = \frac{\partial}{\partial x^\ell} \bar{u}^j(t, x). \quad (4.22)$$

Actually (4.21b) comes from the differentiation of (4.3b) with respect to s , since

$$\partial_\tau \bar{\varphi}^j(\tau; s, y)|_{\tau=s} = \bar{u}^j(s, \bar{\varphi}(s; s, y)) = \bar{u}^j(s, y).$$

Solving (4.21a)(4.21b), we get

$$\bar{\psi}_0(\tau, s, y) = -\exp\left[\int_s^\tau \bar{U}(\tau', \bar{\varphi}(\tau'; s, y)) d\tau'\right] \bar{u}(s, y), \quad (4.23)$$

where $\bar{\psi}_0 = (\bar{\psi}_0^j | j = 1, 2, 3)$ is a 3-dimensional vector and $\bar{U} = (\bar{U}_\ell^j | \ell, j = 1, 2, 3)$ is a 3 by 3 matrix.

Next we consider

$$\bar{\psi}_k^j(t, x) := \frac{\partial}{\partial y^k} \bar{\varphi}^j(\tau; s, y), \quad (4.24)$$

which can evaluate

$$\frac{\partial}{\partial x^j}[\bar{\varphi}^\ell(t_0; t, x)] = \bar{\psi}_j^\ell(\tau, s, y)|_{(\tau, s, y)=(t_0, t, x)}. \quad (4.25)$$

The functions $\bar{\psi}_k^j$ are determined by the initial value problem

$$\frac{\partial}{\partial t} \bar{\psi}_k^j(\tau, s, y) = \sum_{\ell} \bar{U}_{\ell}^j(\tau, \bar{\varphi}(\tau; s, y)) \bar{\psi}_k^{\ell}(\tau, s, y), \quad (4.26a)$$

$$\bar{\psi}_k^j(s, s, y) = \delta_k^j. \quad (4.26b)$$

Actually (4.26b) comes from the differentiation of (4.3b) with respect to y^k . Integrating this, we get

$$\bar{\psi}(\tau, s, y) = \exp \left[\int_s^{\tau} \bar{U}(\tau', \bar{\varphi}(\tau'; s, y)) d\tau' \right], \quad (4.27)$$

where $\bar{\psi} = (\bar{\psi}_k^j | k, j = 1, 2, 3)$ is a 3 by 3 matrix.

Summing up, we see

$$\begin{aligned} C^{\ell}(t, x) &= - \left(\exp \left[\int_t^{t_0} \bar{U}(\tau', \bar{\varphi}(\tau'; t, x)) d\tau' \right] \bar{u}(t, x) \right)^{\ell} + \\ &\quad + \sum_j \bar{u}^j(t, x) \left(\exp \left[\int_t^{t_0} \bar{U}(\tau', \bar{\varphi}(\tau'; t, x)) d\tau' \right] \bar{u}(t, x) \right)_j^{\ell} \\ &= 0 \end{aligned}$$

The proof of the fact $\clubsuit = 0$ is complete, and we can claim that the identity (4.7) holds for any quantity Q .

Next, let us observe the commutation of the operators Δ and the differentiation $\partial/\partial x^j$. We claim:

Formula 2 *For any quantity Q it holds that*

$$d\Delta Q = (\Delta dQ)J + d\bar{Q}(J - I). \quad (4.28)$$

Here, for quantity A , dA denotes the co-variant vector

$$dA = \left(\frac{\partial A}{\partial x^1} \quad \frac{\partial A}{\partial x^2} \quad \frac{\partial A}{\partial x^3} \right) = \sum_{j=1}^3 \frac{\partial A}{\partial x^j} dx^j \quad (4.29)$$

and we put

$$J = (J_j^k \mid j, k = 1, 2, 3), \quad (4.30)$$

with

$$\begin{aligned} J_j^k(t, x) &= \frac{\partial \xi^k}{\partial x^j} + \delta_j^k = \\ &= \sum_{\ell} \frac{\partial}{\partial x^j} [\bar{\varphi}^{\ell}(t_0; t, x)] \frac{\partial}{\partial y^{\ell}} [\varphi^k(t; t_0, y)]|_{y=\bar{\varphi}(t_0; t, x)}. \end{aligned} \quad (4.31)$$

(Recall (4.14) and (4.15).) Therefore (4.28) means

$$\frac{\partial}{\partial x^j}(\Delta Q) = \sum_k \left(\left(\Delta \frac{\partial Q}{\partial x^k} \right) J_j^k + \frac{\partial \bar{Q}}{\partial x^k} (J_j^k - \delta_j^k) \right), \quad j = 1, 2, 3, \quad (4.32)$$

or in other words,

$$\Delta \frac{\partial Q}{\partial x^k} = \sum_j \left(\frac{\partial}{\partial x^j}(\Delta Q) \cdot (J^{-1})_k^j + \frac{\partial \bar{Q}}{\partial x^j} ((J^{-1})_k^j - \delta_k^j) \right). \quad (4.33)$$

Proof can be done directly. In fact we have

$$\begin{aligned} \Delta \frac{\partial}{\partial x^j} Q(t, x) &= \partial_{z^j} Q(t, z)|_{z=x+\xi} - \partial_{x^j} \bar{Q}(t, x), \\ \left(\frac{\partial}{\partial x^j} \Delta Q \right)(t, x) &= \partial_{z^j} Q(t, z)|_{z=x+\xi} - \partial_{x^j} \bar{Q}(t, x) + \sum_k \frac{\partial \xi^k}{\partial x^j}(t, x) \partial_{z^k} Q(t, z)|_{z=x+\xi} \end{aligned}$$

and

$$\partial_{z^k} Q(t, z)|_{z=x+\xi} = (\Delta \partial_{x^k} Q)(t, x) + \partial_{x^k} \bar{Q}(t, x)$$

by the definition.

Note that, if $\bar{u}^j = 0$ for $\forall j$ and $\bar{\varphi}(t; s, y) = y$ identically, then

$$J_j^k(t, x) = \frac{\partial}{\partial x^j} \varphi(t; t_0, x) = \left(\exp \left[\int_{t_0}^t U(\tau', \varphi(\tau'; t_0, x)) d\tau' \right] \right)_j^k,$$

where

$$U(t, x) = (U_\ell^j(t, x) \mid \ell, j = 1, 2, 3) = \left(\frac{\partial u^j}{\partial x^\ell}(t, x) \mid \ell, j = 1, 2, 3 \right).$$

Therefore, if $\bar{u}^j = 0 \forall j$ and $|\partial u^j / \partial x^\ell| \ll 1 \forall \ell, j$, then $J \doteq I$ and (4.28) says $d\Delta Q \doteq \Delta dQ$.

Finally, the following formula can be easily verified:

Formula 3 For any quantities Q_1, Q_2 , it holds that

$$\Delta(Q_1 \cdot Q_2) = (\Delta Q_1) \cdot \bar{Q}_2 + (\bar{Q}_1 + \Delta Q_1) \cdot (\Delta Q_2). \quad (4.34)$$

Formula 4 For any quantity Q and any smooth function F , it holds that

$$\Delta F(Q) = F(\bar{Q} + \Delta Q) - F(\bar{Q}) = \left(\int_0^1 DF(\bar{Q} + \theta \Delta Q) d\theta \right) \cdot (\Delta Q). \quad (4.35)$$

4.2 Rotating frame with a constant angular velocity

We are going to formulate the Euler-Poisson equations in a frame of reference which is rotating with angular velocity $\vec{\Omega}$ around a point O relative to a Newtonian frame. More precisely, let $(t, x) = (t, x^1, x^2, x^3)$ be a Galilean co-ordinates of an inertial system, and we introduce the rotating co-ordinate system (t, y^1, y^2, y^3) defined by

$$x^1 = (\cos \Omega t)y^1 - (\sin \Omega t)y^2, \quad (4.36a)$$

$$x^2 = (\sin \Omega t)y^1 + (\cos \Omega t)y^2, \quad (4.36b)$$

$$x^3 = y^3, \quad (4.36c)$$

or,

$$y^1 = (\cos \Omega t)x^1 + (\sin \Omega t)x^2, \quad (4.37a)$$

$$y^2 = -(\sin \Omega t)x^1 + (\cos \Omega t)x^2, \quad (4.37b)$$

$$y^3 = x^3. \quad (4.37c)$$

Here Ω is a constant. Thus the co-ordinate system (t, y^1, y^2, y^3) is rotating around the x^3 -axis with the angular velocity Ω . We write $\vec{\Omega} = (0, 0, \Omega)$ in the frame (t, x) , that is,

$$\vec{\Omega} := \Omega \frac{\partial}{\partial x^3}. \quad (4.38)$$

Let us consider a vector field

$$\vec{X}(t) = \sum_{j=1}^3 X^j(t) \frac{\partial}{\partial x^j} = \sum_{k=1}^3 Y^k(t) \frac{\partial}{\partial y^k}. \quad (4.39)$$

Then we have the formula

$$\frac{d}{dt} \vec{X}(t) := \sum_j \frac{dX^j}{dt} \frac{\partial}{\partial x^j} = \sum_k \frac{dY^k}{dt} \frac{\partial}{\partial y^k} + \vec{\Omega} \times \vec{X}(t). \quad (4.40)$$

In fact, by the definition, we see

$$X^1 = (\cos \Omega t)Y^1 - (\sin \Omega t)Y^2,$$

$$X^2 = (\sin \Omega t)Y^1 + (\cos \Omega t)Y^2,$$

$$X^3 = Y^3.$$

Therefore

$$\begin{aligned}
\frac{dX^1}{dt} &= (\cos \Omega t) \frac{dY^1}{dt} - (\sin \Omega t) \frac{dY^2}{dt} + \Omega(-(\sin \Omega t)Y^1 - (\cos \Omega t)Y^2) \\
&= \sum_k \frac{dY^k}{dt} \frac{\partial x^1}{\partial y^k} - \Omega X^2, \\
\frac{dX^2}{dt} &= (\sin \Omega t) \frac{dY^1}{dt} + (\cos \Omega t) \frac{dY^2}{dt} + \Omega((\cos \Omega t)Y^1 - (\sin \Omega t)Y^2) \\
&= \sum_k \frac{dY^k}{dt} \frac{\partial x^2}{\partial y^k} + \Omega X^1, \\
\frac{dX^3}{dt} &= \frac{dY^3}{dt} = \sum_k \frac{dY^k}{dt} \frac{\partial x^3}{\partial y^k} + 0,
\end{aligned}$$

which yields (4.40), since (4.38) yields

$$\vec{\Omega} \times \vec{X} = \Omega \left(-X^2 \frac{\partial}{\partial x^1} + X^1 \frac{\partial}{\partial x^2} \right). \quad (4.41)$$

Applying the formula (4.40) to

$$\vec{X}(t) = \vec{r} := \sum_j x^j \frac{\partial}{\partial x^j} = \sum_k y^k \frac{\partial}{\partial y^k}, \quad (4.42)$$

we have

$$\vec{v} = \sum_j v^j \frac{\partial}{\partial x^j} := \frac{d\vec{r}}{dt} = \sum_j \frac{dx^j}{dt} \frac{\partial}{\partial x^j} = \vec{u} + \vec{\Omega} \times \vec{r}, \quad (4.43)$$

where

$$\vec{u} = \sum_k u^k \frac{\partial}{\partial y^k} := \sum_k \frac{dy^k}{dt} \frac{\partial}{\partial y^k}. \quad (4.44)$$

Note that, if we write (4.43) at every components, we have

$$\begin{aligned}
v^1 &= (\cos \Omega t)u^1 - (\sin \Omega t)u^2 - \Omega x^2, \\
v^2 &= (\sin \Omega t)u^1 + (\cos \Omega t)u^2 + \Omega x^1, \\
v^3 &= u^3.
\end{aligned} \quad (4.45)$$

Moreover, applying the formula (4.40) to $\vec{v} = d\vec{r}/dt$ and \vec{u} , we see

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{d\vec{u}}{dt} + \vec{\Omega} \times \frac{d\vec{r}}{dt} \\
&= \frac{d\vec{u}}{dt} + \vec{\Omega} \times (\vec{u} + \vec{\Omega} \times \vec{r}) \\
&= \sum_k \frac{du^k}{dt} \frac{\partial}{\partial y^k} + \vec{\Omega} \times \vec{u} + \vec{\Omega} \times (\vec{u} + \vec{\Omega} \times \vec{r}) \\
&= \sum_k \frac{du^k}{dt} \frac{\partial}{\partial y^k} + 2\vec{\Omega} \times \vec{u} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}).
\end{aligned} \quad (4.46)$$

This can be written as

$$\sum_j \frac{d^2 x^j}{dt^2} \frac{\partial}{\partial x^j} = \sum_k \frac{d^2 y^k}{dt^2} \frac{\partial}{\partial y^k} + 2\vec{\Omega} \times \left(\sum_k \frac{dy^k}{dt} \frac{\partial}{\partial y^k} \right) + \vec{\Omega} \times \left(\vec{\Omega} \times \left(\sum_k y^k \frac{\partial}{\partial y^k} \right) \right). \quad (4.47)$$

Note that

$$\vec{\Omega} = \Omega \frac{\partial}{\partial x^3} = \Omega \frac{\partial}{\partial y^3}$$

so that

$$\vec{\Omega} \times \left(\sum_k y^k \frac{\partial}{\partial y^k} \right) = \Omega \left(-y^2 \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^2} \right)$$

and so on.

It is well-known that $-2\vec{\Omega} \times \vec{u}$ is called the ‘**deflecting** or **Coriolis acceleration**’, and $-\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is called the ‘**centrifugal acceleration**’. These, multiplied by the mass, are not real forces but merely apparent forces.

The observation so far paraphrases mathematically the discussion developed in pp. 139-140 of the Batchelor’s book [1].

4.3 Euler-Poisson equations in the rotating frame

Let us consider the Euler-Poisson equations both in the inertial frame (t, x^1, x^2, x^3) and in the rotating frame (t, y^1, y^2, y^3) considered in the previous subsection.

The velocity fields in the frame (t, x) is

$$\vec{v}(t, x) = \sum_j v^j(t, x) \frac{\partial}{\partial x^j}, \quad (4.48)$$

and

$$v^j = \frac{dx^j}{dt} \quad (4.49)$$

along the stream line $t \mapsto \vec{r} = \sum_j x^j(t) \frac{\partial}{\partial x^j}$ of the particle passing (t, x) .

The equation of continuity (1.1a) is

$$\frac{D\rho}{Dt} + \rho \sum_j \frac{\partial v^j}{\partial x^j} = 0 \quad (4.50)$$

and the equation of motion (1.1b) is

$$\rho \frac{D\vec{v}}{Dt} + \nabla P = -\rho \nabla \Phi, \quad (4.51)$$

where

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \sum_{\ell} v^{\ell} \frac{\partial\rho}{\partial x^{\ell}}, \quad (4.52)$$

$$\frac{D\vec{v}}{Dt} = \sum_j \left(\frac{\partial v^j}{\partial t} + \sum_{\ell} v^{\ell} \frac{\partial v^j}{\partial x^{\ell}} \right) \frac{\partial}{\partial x^j}. \quad (4.53)$$

We are going to write down the equation of motion (4.51) by dint of the rotating frame (t, y^1, y^2, y^3) .

Consider the stream line $t \mapsto \vec{r}(t) = \sum_j x^j(t) \frac{\partial}{\partial x^j}$ such that $\vec{v} = \frac{d\vec{r}}{dt}$, in which sense we can say

$$\frac{D\vec{v}}{Dt} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}.$$

The formula (4.40) tells us

$$\vec{v} = \vec{u} + \vec{\Omega} \times \vec{r},$$

where

$$\vec{u} = \sum_k u^k \frac{\partial}{\partial y^k} = \sum_k \frac{dy^k}{dt} \frac{\partial}{\partial y^k} \quad (4.54)$$

is the velocity field which appears in the frame (t, y^1, y^2, y^3) .

Remark 5 Note that the velocity field is **not** a vector field on \mathbb{R}^3 in the sense of the theory of differentiable manifolds, that is, it is **not** invariant with respect to the change of co-ordinates.

The formula (4.40) tells us

$$\frac{d\vec{v}}{dt} = \frac{d\vec{u}}{dt} + \vec{\Omega} \times \vec{v}.$$

But the formula (4.40) tells us

$$\begin{aligned} \frac{d\vec{u}}{dt} &= \sum_k \frac{du^k}{dt} \frac{\partial}{\partial y^k} + \vec{\Omega} \times \vec{u} \\ &= \sum_k \left(\frac{\partial u^k}{\partial t} + \sum_{\ell} u^{\ell} \frac{\partial u^k}{\partial y^{\ell}} \right) \frac{\partial}{\partial y^k} + \vec{\Omega} \times \vec{u}. \end{aligned}$$

Let us write

$$\frac{Du^k}{Dt} = \frac{\partial u^k}{\partial t} + \sum_{\ell} u^{\ell} \frac{\partial u^k}{\partial y^{\ell}}. \quad (4.55)$$

Then we can write

$$\frac{d\vec{v}}{dt} = \sum_k \frac{Du^k}{Dt} \frac{\partial}{\partial y^k} + 2(\vec{\Omega} \times \vec{u}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad (4.56)$$

with

$$\vec{u} = \sum_k u^k \frac{\partial}{\partial y^k}, \quad \vec{r} = \sum_k y^k \frac{\partial}{\partial y^k},$$

and u^k 's are considered as functions of (t, y^1, y^2, y^3) . Thus the equation of motion (4.51) can be written

$$\begin{aligned} \rho \left[\sum_k \frac{Du^k}{Dt} \frac{\partial}{\partial y^k} + 2\vec{\Omega} \times \left(\sum_k u^k \frac{\partial}{\partial y^k} \right) + \vec{\Omega} \times \left(\vec{\Omega} \times \left(\sum_k y^k \frac{\partial}{\partial y^k} \right) \right) \right] + \\ + \sum_k \frac{\partial P}{\partial y^k} \frac{\partial}{\partial y^k} = -\rho \sum_k \frac{\partial \Phi}{\partial y^k} \frac{\partial}{\partial y^k}, \end{aligned} \quad (4.57)$$

or, more briefly,

$$\rho \left[\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \right] + \nabla P = -\rho \nabla \Phi. \quad (4.58)$$

Here note that

$$\nabla Q = \sum_j \frac{\partial Q}{\partial x^j} \frac{\partial}{\partial x^j} = \sum_k \frac{\partial Q}{\partial y^k} \frac{\partial}{\partial y^k}$$

for any quantity Q . Actually (y^1, y^2, y^3) is an orthonormal co-ordinate system for each fixed t . The equation (4.58) is nothing but [7, p.295, (13)]. (Here we are writing P, Φ instead of $p, -\psi$ of [7], and neglecting other external force \mathbf{F} .)

Note that it can be easily verified that the vector product performs in the same manner in the frame (t, y^1, y^2, y^3) , that is,

$$\begin{aligned} \frac{\partial}{\partial y^1} \times \frac{\partial}{\partial y^2} &= -\frac{\partial}{\partial y^2} \times \frac{\partial}{\partial y^1} = \frac{\partial}{\partial y^3}, \\ \frac{\partial}{\partial y^2} \times \frac{\partial}{\partial y^3} &= -\frac{\partial}{\partial y^3} \times \frac{\partial}{\partial y^2} = \frac{\partial}{\partial y^1}, \\ \frac{\partial}{\partial y^3} \times \frac{\partial}{\partial y^1} &= -\frac{\partial}{\partial y^1} \times \frac{\partial}{\partial y^3} = \frac{\partial}{\partial y^2}, \\ \frac{\partial}{\partial y^1} \times \frac{\partial}{\partial y^1} &= \frac{\partial}{\partial y^2} \times \frac{\partial}{\partial y^2} = \frac{\partial}{\partial y^3} \times \frac{\partial}{\partial y^3} = 0. \end{aligned} \quad (4.59)$$

On the other hand, by tedious calculations, it can be verified that

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\frac{1}{2} \nabla (|\vec{\Omega} \times \vec{r}|^2). \quad (4.60)$$

Here

$$|\vec{\Omega} \times \vec{r}|^2 = (\Omega^2 x^3 - \Omega^3 x^2)^2 + (\Omega^3 x^1 - \Omega^1 x^3)^2 + (\Omega^1 x^2 - \Omega^2 x^1)^2$$

for $\vec{\Omega} = \sum_j \Omega^j \frac{\partial}{\partial x^j}$, $\vec{r} = \sum_j x^j \frac{\partial}{\partial x^j}$. Using this identity (4.60), the equation (4.58) can be written as

$$\rho \left(\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} \right) + \nabla P = -\rho \nabla \left(\Phi - \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 \right). \quad (4.61)$$

This is nothing but [6, p.500, (1)].

Note that this formulation of the equation of motion, in which the centrifugal acceleration is considered as a part of the potential, can be traced back to the paper [2] by V. Bjerknes in 1929. See [2, p.11, (A)].

On the other hand, by a direct calculation, it can be verified that the equation of continuity (4.50), which can be written as

$$\frac{D\rho}{Dt} + \rho(\nabla|\vec{v}) = 0,$$

is written down in the rotating frame as the apparently same form

$$\frac{D\rho}{Dt} + \rho(\nabla|\vec{u}) = 0. \quad (4.62)$$

Here we are denoting

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \sum_k u^k \frac{\partial\rho}{\partial y^k} \quad (4.63)$$

and

$$(\nabla|\vec{u}) = \sum_k \frac{\partial u^k}{\partial y^k} \quad (4.64)$$

More precisely speaking, (4.63) should be written as

$$\frac{D\rho}{Dt} = \left(\frac{\partial\rho}{\partial t} \right)_y + \sum_k u^k \frac{\partial\rho}{\partial y^k},$$

where the symbol $(\partial/\partial t)_y$ emphasizes that the partial differentiation with respect to t is done keeping y constant. In this symbol, (4.52) means

$$\frac{D\rho}{Dt} = \left(\frac{\partial\rho}{\partial t} \right)_x + \sum_\ell v^\ell \frac{\partial\rho}{\partial x^\ell}.$$

Note that

$$\left(\frac{\partial\rho}{\partial t} \right)_x \neq \left(\frac{\partial\rho}{\partial t} \right)_y, \quad \sum_j v^j \frac{\partial\rho}{\partial x^j} \neq \sum_k u^k \frac{\partial\rho}{\partial y^k}$$

generally if $\Omega \neq 0$.

However, as for (4.64), it can be verified by (4.45) that

$$\sum_j \frac{\partial v^j}{\partial x^j} = \sum_k \frac{\partial u^k}{\partial y^k}, \quad (4.65)$$

although it is not obvious. We might write the left-hand side of (4.65) as $(\nabla_x|\vec{v})$ and the right-hand side as $(\nabla_y|\vec{u})$, and the equality as

$$(\nabla_x|\vec{v}) = (\nabla_y|\vec{u}).$$

As for the Poisson equation and the Newton potential the expression in the frame (t, y^1, y^2, y^3) is the same as those in (t, x^1, x^2, x^3) , since the co-ordinate transformation $(x^1, x^2, x^3) \mapsto (y^1, y^2, y^3)$ is orthonormal for each fixed t .

In fact the determinant of the Jacobian matrix

$$\begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \frac{\partial x^1}{\partial y^3} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \frac{\partial x^2}{\partial y^3} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} & \frac{\partial x^3}{\partial y^3} \end{bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is equal to 1 so that

$$d\mathcal{V} = dx^1 dx^2 dx^3 = dy^1 dy^2 dy^3,$$

and it can be verified easily that

$$|\vec{r}|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 = (y^1)^2 + (y^2)^2 + (y^3)^2,$$

$$\Delta \Phi = \sum_j \left(\frac{\partial}{\partial x^j} \right)^2 \Phi = \sum_k \left(\frac{\partial}{\partial y^k} \right)^2 \Phi.$$

4.4 Perturbation of the Euler-Poisson equations in the rotating frame

Let us observe the perturbation of the Euler-Poisson equations in the rotating frame considered in the preceding subsections. **However let us write (t, x^1, x^2, x^3) instead of (t, y^1, y^2, y^3) , and $\vec{v} = (v^1, v^2, v^3)$ instead of $\vec{u} = (u^1, u^2, u^3)$.** Therefore the Euler-Poisson equations in this rotating frame (t, x^1, x^2, x^3) are

$$\frac{D\rho}{Dt} + \rho(\nabla|\vec{v}) = 0, \tag{4.66a}$$

$$\rho \left[\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \right] + \nabla P = -\rho \nabla \Phi, \tag{4.66b}$$

$$\Delta \Phi = 4\pi G \rho, \tag{4.66c}$$

where

$$\begin{aligned}\frac{DQ}{Dt} &= \frac{\partial Q}{\partial t} + \sum_{k=1}^3 v^k(t, x) \frac{\partial Q}{\partial x^k} \quad \text{for } Q = \rho, v^j, \\ (\nabla|\vec{v}) &= \sum_{k=1}^3 \frac{\partial v^k}{\partial x^k}, \quad \vec{\Omega} = \Omega \frac{\partial}{\partial x^3}, \quad \vec{r} = \sum_{j=1}^3 x^j \frac{\partial}{\partial x^j}, \\ \nabla Q &= \sum_{j=1}^3 \frac{\partial Q}{\partial x^j} \frac{\partial}{\partial x^j} \quad \text{for } Q = P, \Phi,\end{aligned}$$

and

$$\Delta \Phi = \sum_{j=1}^3 \left(\frac{\partial}{\partial x^j} \right)^2 \Phi.$$

Supposing that the support of $\rho(t, \cdot)$ is compact, we replace the Poisson equation (4.66c) by the Newton potential

$$\Phi(t, x) = -G \int \int \int \frac{\rho(t, x')}{|x - x'|} dx'_1 dx'_2 dx'_3. \quad (4.67)$$

Let us consider the unperturbed motion which is static in this rotating frame, that is, $\bar{\rho} = \bar{\rho}(x), \bar{u}^j = 0 \ \forall j, \ \bar{\Phi} = \bar{\Phi}(x)$. Of course the equation

$$\bar{\rho} \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \nabla \bar{P} = -\bar{\rho} \nabla \bar{\Phi} \quad (4.68)$$

should hold.

In this case, since $\bar{\varphi}(t; t_0, x) = x$ identically, we have

$$x^j + \xi^j(t, x) = \varphi^j(t; t_0, x). \quad (4.69)$$

So, by the definition, we have

$$\Delta \vec{r} = \sum_j \xi^j(t, x) \frac{\partial}{\partial x^j}, \quad (4.70)$$

$$\Delta v^j(t, x) = v^j(t, x + \xi(t, x)) = \frac{\partial}{\partial t} \xi^j(t, x), \quad (4.71)$$

and Formula 1 implies

$$\left(\Delta \frac{Dv^j}{Dt} \right)(t, x) = \frac{\partial^2 \xi^j}{\partial t^2}(t, x). \quad (4.72)$$

Let us observe the equation of continuity (4.66a). Integration of this equation is

$$\rho(t, x + \xi) = \rho(t_0, x) \exp \left[- \int_{t_0}^t \text{tr} V(\tau, \varphi(\tau; t_0, x)) d\tau \right], \quad (4.73)$$

where

$$V(t, x) = \left(\frac{\partial v^j}{\partial x^k}(t, x) \mid k, j = 1, 2, 3 \right), \quad \text{and} \quad \text{tr} V = \sum_j \frac{\partial v^j}{\partial x^j} \quad (4.74)$$

But we can evaluate

$$\frac{\partial}{\partial z^\ell} [v^j(t, z)] \Big|_{z=x+\xi} (= V_\ell^j(t, x + \xi)) = \sum_k \left(\frac{\partial J}{\partial t} \right)_k^j (J^{-1})_\ell^k(t, x), \quad (4.75)$$

where the matrix $J = (J_k^j \mid k, j = 1, 2, 3)$ is defined by

$$J_k^j(t, x) = \delta_k^j + \frac{\partial \xi^j}{\partial x^k}, \quad (4.76)$$

and $J^{-1}(t, x)$ is the inverse matrix of $J(t, x)$. In fact, differentiating (4.69), we get

$$\begin{aligned} \frac{\partial}{\partial t} \xi^j(t, x) &= \frac{\partial}{\partial t} \varphi^j(t; t_0, x) = v^j(t, x + \xi(t, x)), \\ \frac{\partial^2}{\partial t \partial x^k} \xi^j(t, x) &= \sum_\ell \frac{\partial}{\partial z^\ell} v^j(t, z) \Big|_{z=x+\xi} \cdot \left(\delta_k^\ell + \frac{\partial \xi^\ell}{\partial x^k} \right), \end{aligned}$$

that is,

$$\frac{\partial}{\partial t} J_k^j = \sum_\ell V_\ell^j(t, x + \xi(t, x)) J_k^\ell.$$

Therefore, since

$$\text{tr} V(t, \varphi(t; t_0, x)) = \text{tr} \left(\frac{\partial J}{\partial t} J^{-1} \right)(t, x) = \frac{\partial}{\partial t} \log \det J(t, x)$$

and since $J(t_0, x) = I$ (the unit matrix), we see that (4.73) reads

$$\rho(t, x + \xi(t, x)) = \bar{\rho}(x) + \Delta \rho(t, x) = \overset{\circ}{\rho}(x) \det J(t, x)^{-1}, \quad (4.77)$$

where $\overset{\circ}{\rho}(x) = \rho(t_0, x)$ is initial data.

Let us assume that

$$\mathcal{D} := \{x \in \mathbb{R}^3 \mid \bar{\rho}(x) > 0\} \quad (4.78)$$

is a simply connected domain including the origin O . Put

$$q(t, x) := \frac{(\Delta \rho)(t, x)}{\bar{\rho}(x)} \quad (4.79)$$

for $x \in \mathcal{D}$, or in other words,

$$\rho(t, x + \xi(t, x)) = \bar{\rho}(x) + \Delta\rho(t, x) = \bar{\rho}(x)(1 + q(t, x)). \quad (4.80)$$

We are supposing that the relative perturbation $q(t, x)$ is small. Then the equation (4.77) reads

$$q(t, x) = -1 + (1 + \overset{\circ}{q}(x)) \det J(t, x)^{-1}, \quad (4.81)$$

where $\overset{\circ}{q}(x) = q(t_0, x)$ is initial data.

Let us linearize the equation (4.77), that is, **hereafter we shall denote $Q \cong 0$ if Q is of higher order than second of $\Delta\rho, v^j, \xi^j$ and their derivatives.**

Since

$$\det J(t, x)^{-1} \cong 1 - \sum_j \frac{\partial \xi^j}{\partial x^j}, \quad (4.82)$$

the linearized approximation of the equation (4.77) is

$$(\Delta\rho)(t, x) \cong (\Delta\rho)(t_0, x) - \bar{\rho}(x) \sum_j \frac{\partial \xi^j}{\partial x^j}(t, x), \quad (4.83)$$

This means that, once the evolution of Lebovitz' displacement $\xi(t, x)$ has been solved, then the evolution of the Lagrangian change $\Delta\rho$ of the density is determined, in the sense of linearized approximation, by the divergence of $\xi(t, x)$ provided that the initial perturbation $\Delta\rho(t_0, x) = \rho(t_0, x) - \bar{\rho}(x)$ is given.

Remark 6 *I do not see why D. Lynden-Bell and J. P. Ostriker can claim [7, p.296, (19)], that is,*

$$\Delta\rho + \bar{\rho} \sum_j \frac{\partial \xi^j}{\partial x^j} = 0,$$

even if the initial perturbation $\Delta\rho(t_0, x)$ does not identically vanish.

The linearized approximation (4.81) says

$$q(t, x) \cong \overset{\circ}{q}(x) - \sum_j \frac{\partial \xi^j}{\partial x^j}(t, x), \quad (4.84)$$

for $x \in \mathcal{D}$.

Now, operating Δ on the equation of motion (4.66b) yields

$$\begin{aligned} (\Delta\rho)\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + (\bar{\rho} + \Delta\rho) \left[\Delta \frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \Delta\vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \Delta\vec{r}) \right] + \\ + \Delta\nabla P = -(\Delta\rho)(\nabla\bar{\Phi}) - (\bar{\rho} + \Delta\rho)\Delta\nabla\Phi \end{aligned} \quad (4.85)$$

by Formula 3. On the domain \mathcal{D} , this can be written as

$$\begin{aligned} q \cdot \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + (1+q) \left[\frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right] + \\ + \frac{1}{\bar{\rho}} \Delta \nabla P = -q(\nabla \bar{\Phi}) - (1+q) \Delta \nabla \Phi. \end{aligned} \quad (4.86)$$

Here we have used (4.70), (4.71), (4.72), and, of course,

$$\frac{\partial^2 \vec{\xi}}{\partial t^2}, \quad \frac{\partial \vec{\xi}}{\partial t}, \quad \vec{\xi}$$

mean

$$\sum_j \frac{\partial^2 \xi^j}{\partial t^2}(t, x) \frac{\partial}{\partial x^j}, \quad \sum_j \frac{\partial \xi^j}{\partial t}(t, x) \frac{\partial}{\partial x^j}, \quad \sum_j \xi^j(t, x) \frac{\partial}{\partial x^j}$$

respectively. Using (4.68), which says

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \frac{1}{\bar{\rho}} \nabla \bar{P} = -\nabla \bar{\Phi}$$

on \mathcal{D} , we have the equation of motion:

$$(1+q) \left[\frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right] - \frac{q}{\bar{\rho}} \nabla \bar{P} + \frac{1}{\bar{\rho}} \Delta \nabla P = -(1+q) \Delta \nabla \Phi, \quad (4.87)$$

which should hold on the domain $[0, T] \times \mathcal{D}$.

The linearized approximation of this equation is

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) - \frac{q}{\bar{\rho}} \nabla \bar{P} + \frac{1}{\bar{\rho}} \Delta \nabla P \cong -\Delta \nabla \Phi. \quad (4.88)$$

Now we are going to analyze the term

$$-\frac{q}{\bar{\rho}} \nabla \bar{P} + \frac{1}{\bar{\rho}} \Delta \nabla P.$$

In order to do it, let us introduce the state variable, enthalpy, u defined by

$$u = \int \frac{dP}{\rho} = \frac{A\gamma}{\gamma-1} \rho^{\gamma-1}, \quad (4.89)$$

supposing the equation of state is

$$P = A\rho^\gamma, \quad (4.90)$$

where A, γ are positive constants such that $1 < \gamma < 2$. Then we have

$$\begin{aligned} \Delta u = u(t, x + \xi) - \bar{u}(x) &= \frac{A\gamma}{\gamma-1} \left(\rho(t, x + \xi)^{\gamma-1} - \bar{\rho}(x)^{\gamma-1} \right) = \\ &= \bar{u}(x) ((1+q)^{\gamma-1} - 1). \end{aligned} \quad (4.91)$$

So, using Formula 2, 3, we see

$$\begin{aligned}
-q\frac{d\bar{P}}{\bar{\rho}} + \frac{1}{\bar{\rho}}\Delta dP &= -qd\bar{u} + \frac{1}{\bar{\rho}}\Delta(\rho du) \\
&= -qd\bar{u} + \frac{1}{\bar{\rho}}((\Delta\rho)d\bar{u} + (\bar{\rho} + \Delta\rho)\Delta u) \\
&= (1+q)\Delta du \\
&= (1+q)((d\Delta u)J^{-1} + (d\bar{u})(J^{-1} - I)) \\
&= (1+q)\left(d(\bar{u}((1+q)^{\gamma-1} - 1))J^{-1} + (d\bar{u})(J^{-1} - I)\right). \quad (4.92)
\end{aligned}$$

That is,

$$\begin{aligned}
-q\frac{1}{\bar{\rho}}\partial_{x^j}\bar{P} + \frac{1}{\bar{\rho}}\Delta\partial_{x^j}P &= \\
&= (1+q)\left(\sum_k \frac{\partial}{\partial x^k}\left[\bar{u}((1+q)^{\gamma-1} - 1)\right](J^{-1})_j^k + \sum_k \frac{\partial\bar{u}}{\partial x^k}((J^{-1})_j^k - \delta_j^k)\right) \quad (4.93)
\end{aligned}$$

$$\cong (\gamma - 1)\frac{\partial}{\partial x^j}(\bar{u}q) - \sum_k \frac{\partial\bar{u}}{\partial x^k}\frac{\partial\xi^k}{\partial x^j}. \quad (4.94)$$

Remark 7 *It is easy to see that the conclusion (4.94) coincides with that of [7, p.297, (25)(26)], provided that $\overset{\circ}{\rho} = \bar{\rho}$ so that $\overset{\circ}{q} = 0$ and $q \cong -\sum_k \partial\xi^k/\partial x^k$, keeping in mind that*

$$du = \frac{dP}{\rho} \quad \text{and} \quad \frac{P}{\rho} = \frac{\gamma-1}{\gamma}u.$$

Finally we analyze $\Delta\nabla\Phi$, at the fixed time t .

Let us define linear operators \mathcal{K} and $\mathcal{K}_{,j}$, which act on compactly supported continuous functions f on \mathbb{R}^3 , by

$$(\mathcal{K}f)(x) = \frac{1}{4\pi} \int \frac{f(x')}{|x-x'|} d\mathcal{V}(x'), \quad (4.95)$$

$$(\mathcal{K}_{,j}f)(x) = -\frac{1}{4\pi} \int \frac{(x-x')^j}{|x-x'|^3} f(x') d\mathcal{V}(x'), \quad j = 1, 2, 3. \quad (4.96)$$

Then the Poisson equation

$$\Delta\Phi = 4\pi\mathbb{G}\rho$$

is solved by the Newton potential

$$\Phi(x) = -4\pi\mathbb{G}(\mathcal{K}\rho)(x),$$

provided that the support of ρ is compact, and we have

$$\frac{\partial}{\partial x^j} \Phi(x) = -4\pi \mathbf{G}(\mathcal{K}_{,j} \rho)(x).$$

Let us find the linearized approximation of

$$\left(\Delta \frac{\partial \Phi}{\partial x^j} \right)(t, x) = \frac{\partial}{\partial z^j} \Phi(t, z) \Big|_{z=x+\xi(t,x)} - \frac{\partial \Phi}{\partial x^j}(x).$$

Now the term

$$\frac{\partial}{\partial z^j} \Phi(t, z) \Big|_{z=x+\xi(t,x)} = -4\pi \mathbf{G} \mathcal{K}_{,j} \rho(t, x + \xi(t, x))$$

can be analyzed by

$$\begin{aligned} \mathcal{K}_{,j} \rho(t, x + \xi(t, x)) &= -\frac{1}{4\pi} \int \frac{(x + \xi(t, x) - z')^j}{|x + \xi(t, x) - z'|^3} \rho(z') d\mathcal{V}(z') \\ &= -\frac{1}{4\pi} \int \frac{(x + \xi(t, x) - z')^j}{|x + \xi(t, x) - z'|^3} \mathring{\rho}(x') d\mathcal{V}(x'), \end{aligned}$$

where we put $z' = x' + \xi(t, x')$ and we have used (4.77) to change the variable of integration.

Denoting $z = x + \xi(t, x)$, this is nothing but

$$\mathcal{K}_{,j} \rho(t, x + \xi(t, x)) = \frac{1}{4\pi} \int \frac{\partial}{\partial z^j} \left(\frac{1}{|z - z'|} \right) \mathring{\rho}(x') d\mathcal{V}(x').$$

Let us consider the linearized approximation of this term.

We see

$$\begin{aligned} \frac{\partial}{\partial z^j} \left(\frac{1}{|z - z'|} \right) &\cong \frac{\partial}{\partial x^j} \left(\frac{1}{|x - x'|} \right) + \sum_k \frac{\partial^2}{\partial x^k \partial x^j} \left(\frac{1}{|x - x'|} \right) \cdot \xi^k(t, x) + \\ &+ \sum_k \frac{\partial^2}{\partial x'^k \partial x^j} \left(\frac{1}{|x - x'|} \right) \xi^k(t, x') \end{aligned}$$

for $z = x + \xi(t, x)$, $z' = x' + \xi(t, x')$. Therefore we have

$$\mathcal{K}_{,j} \rho(t, z) \cong \mathcal{K}_{,j} \mathring{\rho} + \sum_k \left(\frac{\partial}{\partial x^k} \mathcal{K}_{,j} \mathring{\rho} \right) \cdot \xi^k - \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} \mathring{\rho} \xi^k \right],$$

where we have used the integration by parts in the last term.

Thus we get the linearized approximation

$$\Delta \frac{\partial \Phi}{\partial x^j} \cong \frac{\partial \mathring{\Phi}}{\partial x^j} + \sum_k \left(\frac{\partial^2 \mathring{\Phi}}{\partial x^k \partial x^j} \right) \cdot \xi^k + 4\pi \mathbf{G} \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} (\mathring{\rho} \xi^k) \right] - \frac{\partial \bar{\Phi}}{\partial x^j}. \quad (4.97)$$

I owe the above observation of the linearized perturbation of the gravitational potential to Juhi Jang [4].

Remark 8 Suppose that $\overset{\circ}{\rho} = \bar{\rho}$. Then (4.97) says

$$\Delta \frac{\partial \Phi}{\partial x^j} \cong \sum_x \frac{\partial^2 \bar{\Phi}}{\partial x^k \partial x^j} \xi^k + 4\pi G \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} (\bar{\rho} \xi^k) \right]. \quad (4.98)$$

On the other hand [7, (22)(23)] says

$$\begin{aligned} \Delta \frac{\partial \Phi}{\partial x^j} &\cong \sum_k \frac{\partial^2 \bar{\Phi}}{\partial x^k \partial x^j} \xi^k - 4\pi G \frac{\partial}{\partial x^j} \int \frac{\bar{\rho}(x')}{|x - x'|} (\xi(t, x') | d\mathcal{S}(x')) \\ &\quad + 4\pi G \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} (\bar{\rho} \xi^k) \right], \end{aligned} \quad (4.99)$$

where $d\mathcal{S}(x')$ is the ‘outward normal vectorial surface element of the unperturbed fluid’ ([7, p.296, line.8 from below]). However, what is the integral surface of this term (the second term of the right-hand side of (1.4))? If \mathcal{D} is the domain occupied by the unperturbed density $\bar{\rho}$, the integral surface should be $\partial\mathcal{D}$ or larger one outside the support $\mathcal{D}^a = \mathcal{D} \cup \partial\mathcal{D}$. But there $\bar{\rho}$ vanishes. If so, this term is negligible, and (4.99) coincides with (4.98). Here it seems that the discussion of [7] contains confusion due to formal manipulation of Green’s formula applied to

$$\begin{aligned} \delta\Phi(t, x) &= -G \int \left(\frac{1}{|x - (x' + \xi(t, x'))|} - \frac{1}{|x - x'|} \right) \bar{\rho}(x') d\mathcal{V}(x') \\ &\cong -G \int \left(\xi(t, x') \left| \nabla_{x'} \frac{1}{|x - x'|} \right| \right) \bar{\rho}(x') d\mathcal{V}(x'). \end{aligned}$$

Summing up, we get the linearized approximation of (4.87) for the unknown functions $\xi^j(t, x)$, $j = 1, 2, 3$, as follows:

$$\begin{aligned} &\frac{\partial^2 \xi^j}{\partial t^2} + \left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^j \\ &\quad + (\gamma - 1) \frac{\partial}{\partial x^j} (\bar{u} \overset{\circ}{q}) - (\gamma - 1) \frac{\partial}{\partial x^j} \left(\bar{u} \sum_k \frac{\partial \xi^k}{\partial x^k} \right) - \sum_k \frac{\partial \bar{u}}{\partial x^k} \frac{\partial \xi^k}{\partial x^j} + \\ &\quad + \frac{\partial \overset{\circ}{\Phi}}{\partial x^j} + \sum_k \left(\frac{\partial^2 \overset{\circ}{\Phi}}{\partial x^k \partial x^j} \right) \cdot \xi^k + 4\pi G \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} (\overset{\circ}{\rho} \xi^k) \right] - \frac{\partial \bar{\Phi}}{\partial x^j} \cong 0. \end{aligned} \quad (4.100)$$

Here $\left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^j$ are given by

$$\left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^1 = \Omega \left(-2 \frac{\partial \xi^2}{\partial t} - \Omega \xi^1 \right), \quad (4.101a)$$

$$\left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^2 = \Omega \left(2 \frac{\partial \xi^1}{\partial t} - \Omega \xi^2 \right), \quad (4.101b)$$

$$\left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^3 = 0. \quad (4.101c)$$

Note that the right-hand sides of the linearized equations do not contain time derivatives of the unknown functions ξ^j 's.

Moreover, when we neglect $\overset{\circ}{q} = (\overset{\circ}{\rho} - \bar{\rho})/\bar{\rho}$, then (4.100) reduces to

$$\begin{aligned} & \frac{\partial^2 \xi^j}{\partial t^2} + \left(2\vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\xi}) \right)^j \\ & - (\gamma - 1) \frac{\partial}{\partial x^j} \left(\bar{u} \cdot \sum_k \frac{\partial \xi^k}{\partial x^k} \right) - \sum_k \frac{\partial \bar{u}}{\partial x^k} \frac{\partial \xi^k}{\partial x^j} + \\ & + \sum_k \left(\frac{\partial^2 \bar{\Phi}}{\partial x^k \partial x^j} \right) \cdot \xi^k + 4\pi G \frac{\partial}{\partial x^j} \mathcal{K} \left[\sum_k \frac{\partial}{\partial x^k} (\bar{\rho} \xi^k) \right] \cong 0. \end{aligned} \quad (4.102)$$

5 Lagrangian variation and rotating frame for axisymmetric case

5.1 Axisymmetric solutions in the rotating frame

Let us derive the equations for axisymmetric solutions of the equations (4.66a) (4.66b) in the rotating frame (t, x^1, x^2, x^3) with the angular velocity $\vec{\Omega} = \Omega \partial / \partial x^3$. We are using the co-ordinate system (r, ζ, ϕ) defined by

$$x^1 = r\sqrt{1 - \zeta^2} \cos \phi, \quad x^2 = r\sqrt{1 - \zeta^2} \sin \phi, \quad x^3 = r\zeta. \quad (5.1)$$

The density ρ and the gravitational potential Φ is supposed to be functions of (t, r, ζ) and the velocity field $\vec{v} = \sum_j v^j \partial / \partial x^j$ is supposed to be of the form

$$v^1 = \frac{x^1}{r} v - \frac{x^1 \zeta}{1 - \zeta^2} w - x^2 \omega, \quad (5.2a)$$

$$v^2 = \frac{x^2}{r} v - \frac{x^2 \zeta}{1 - \zeta^2} w + x^1 \omega, \quad (5.2b)$$

$$v^3 = \zeta v + r w, \quad (5.2c)$$

where v, w, ω are functions of (t, r, ζ) .

Then, by the same calculation as that in §2.2, we see that the equations (4.66a)(4.66b) reduce to

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial v}{\partial r} + \frac{2}{r} v + \frac{\partial w}{\partial \zeta} \right) = 0, \quad (5.3a)$$

$$\rho \left(\frac{Dv}{Dt} - \frac{r}{1 - \zeta^2} w^2 - r(1 - \zeta^2)(\Omega + \omega)^2 \right) + \frac{\partial P}{\partial r} + \rho \frac{\partial \Phi}{\partial r} = 0, \quad (5.3b)$$

$$\rho \left(\frac{Dw}{Dt} + \frac{2}{r} v w + \frac{\zeta}{1 - \zeta^2} w^2 + \zeta(1 - \zeta^2)(\Omega + \omega)^2 \right) + \frac{1 - \zeta^2}{r^2} \left(\frac{\partial P}{\partial \zeta} + \rho \frac{\partial \Phi}{\partial \zeta} \right) = 0, \quad (5.3c)$$

$$\rho \left(\frac{D\omega}{Dt} + \frac{2}{r} v(\Omega + \omega) - \frac{2\zeta}{1 - \zeta^2} w(\Omega + \omega) \right) = 0, \quad (5.3d)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial \zeta}. \quad (5.4)$$

Note that, as observed in Remark 2, the equation (5.3d) reduces to

$$\rho \frac{D}{Dt} \left(r^2 (1 - \zeta^2) (\Omega + \omega) \right) = 0, \quad (5.5)$$

which can be integrated immediately once v, w are supposed to be known, provided that the initial data $\overset{\circ}{\omega}(r, \zeta) = \omega(t_0, r, \zeta)$ is given. Then the equations (5.3a)(5.3b)(5.3c) turn out to be a closed integro-differential evolution system for unknown ρ, v, w , provided that $\Phi, \overset{\circ}{\omega}$ are given.

Of course the Neewton potential is given by

$$\Phi(t, r, \zeta) = -G \int_{-1}^1 \int_0^\infty K_{II}(r, \zeta, r', \zeta') \rho(t, r', \zeta') r'^2 dr' d\zeta', \quad (5.6)$$

where

$$K_{II}(r, \zeta, r', \zeta') = \int_0^{2\pi} \frac{d\beta}{\sqrt{r^2 + r'^2 - 2rr'(\sqrt{1 - \zeta^2}\sqrt{1 - \zeta'^2} \cos \beta + \zeta\zeta')}}. \quad (5.7)$$

5.2 Axisymmetric perturbation in the rotating frame

Let us consider the axisymmetric perturbation in the rotating frame.

In order to do it, we introduce the notation of the co-ordinate system $(t, y) = (t, y^1, y^2, y^3)$ defined by $y^1 = r, y^2 = \zeta, y^3 = \phi$, while $(t, x) = (t, x^1, x^2, x^3)$ is related as

$$x^1 = r\sqrt{1 - \zeta^2} \cos \phi, \quad x^2 = r\sqrt{1 - \zeta^2} \sin \phi, \quad x^3 = r\zeta.$$

We are fixing a stationary solution $\bar{\rho} = \bar{\rho}(r, \zeta), \bar{v} = \bar{w} = \bar{\omega} = 0, \bar{\Phi} = \bar{\Phi}(r, \zeta)$ of (5.3a)(5.3b)(5.3c)(5.3d)(5.6).

Let us denote by $\eta = (\eta^1, \eta^2, \eta^3)$ the Lebovitz' displacement with respect to the co-ordinate system (t, y^1, y^2, y^3) , that is: Let $t \mapsto y = \psi(t; s, z) = (\psi^j(t; s, z))_{j=1,2,3}$ be the flow passing $(t, y) = (s, z)$, that is,

$$\begin{aligned} \frac{\partial}{\partial t} \psi^j(t; s, z) &= u^j(t, \psi(t; s, z)), \\ \psi^j(s; s, z) &= z^j, \end{aligned}$$

where $\vec{u}(t, y) = (u^j(t, y)|j = 1, 2, 3)$ is the velocity field defined by

$$u^1(t, y) = v(t, r, \zeta), \quad u^2(t, y) = w(t, r, \zeta), \quad u^3(t, y) = \omega(t, r, \zeta)$$

for $y = (r, \zeta, \phi)$; Put

$$\eta^j(t, y) := \psi^j(t; t_0, y) - y^j, \quad j = 1, 2, 3.$$

Since $y \mapsto u^j(t, y)$ are axially symmetric, it is easy to show that $\psi^j(t; 0, y)$ are of the form

$$\psi^1(t; t_0, y) = \check{\psi}^1(t, r, \zeta), \quad \psi^2(t; t_0, y) = \check{\psi}^2(t, r, \zeta), \quad \psi^3(t; t_0, y) = \check{\psi}^3(t, r, \zeta) + \phi,$$

where $y = (r, \zeta, \phi)$. This implies that $\eta^j(t, y)$ are of the form

$$\eta^j(t, y) = \check{\eta}^j(t, r, \zeta), \quad j = 1, 2, 3,$$

where $y = (r, \zeta, \phi)$. In other words, the Lebovitz' displacement is axially symmetric, too. Hereafter we shall denote η for $\check{\eta}$ because no confusions might occur.

The Lebovitz' change operator Δ is of course defined by

$$\Delta Q(t, y) = Q(t, y + \eta(t, y)) - \bar{Q}(t, y).$$

Then we see

$$\Delta r = \eta^1, \quad \Delta \zeta = \eta^2, \quad \Delta \phi = \eta^3,$$

and

$$\Delta u^j(t, y) = u^j(t, y + \eta(t, y)) = \frac{\partial \eta^j}{\partial t}(t, y), \quad j = 1, 2, 3,$$

that is,

$$\Delta v = \frac{\partial \eta^1}{\partial t}, \quad \Delta w = \frac{\partial \eta^2}{\partial t}, \quad \Delta \omega = \frac{\partial \eta^3}{\partial t},$$

and

$$\Delta \frac{Du^j}{Dt}(t, y) = \frac{\partial^2 \eta^j}{\partial t^2}(t, y), \quad j = 1, 2, 3,$$

that is,

$$\Delta \frac{Dv}{Dt} = \frac{\partial^2 \eta^1}{\partial t^2}, \quad \Delta \frac{Dw}{Dt} = \frac{\partial^2 \eta^2}{\partial t^2}, \quad \Delta \frac{D\omega}{Dt} = \frac{\partial^2 \eta^3}{\partial t^2}.$$

Remark 9 *However note that*

$$\begin{aligned} \sum_k x^k \frac{\partial}{\partial x^k} &= r \frac{\partial}{\partial r} - \sin(2\phi) \frac{\partial}{\partial \phi} \\ &\neq \sum_j y^j \frac{\partial}{\partial y^j} = r \frac{\partial}{\partial r} + \zeta \frac{\partial}{\partial \zeta} + \phi \frac{\partial}{\partial \phi}. \end{aligned}$$

Operating Δ on the equation (5.3a) results

$$\frac{\partial}{\partial t}(\Delta\rho) + (\bar{\rho} + \Delta\rho)\Delta\left[\frac{\partial v}{\partial r} + \frac{2}{r}v + \frac{\partial w}{\partial \zeta}\right] = 0. \quad (5.8)$$

Here $\Delta\left[\frac{\partial v}{\partial r} + \frac{2}{r}v + \frac{\partial w}{\partial \zeta}\right]$ is a function of derivatives of η^1, η^2 . In fact, putting

$$\mathcal{J} = \begin{bmatrix} 1 + \frac{\partial \eta^1}{\partial r} & \frac{\partial \eta^1}{\partial \zeta} \\ \frac{\partial \eta^2}{\partial r} & 1 + \frac{\partial \eta^2}{\partial \zeta} \end{bmatrix}, \quad (5.9)$$

we see

$$\frac{\partial}{\partial t}\mathcal{J}(t, r, \zeta) = \begin{bmatrix} \left(\frac{\partial v}{\partial r}\right)^* & \left(\frac{\partial v}{\partial \zeta}\right)^* \\ \left(\frac{\partial w}{\partial r}\right)^* & \left(\frac{\partial w}{\partial \zeta}\right)^* \end{bmatrix} \mathcal{J}(t, r, \zeta),$$

where $Q^*(t, y) := Q(t, y + \eta(t, y))$. (Note that $Q^* = \Delta Q$ if $\bar{Q} = 0$.)

Therefore we have

$$\Delta\frac{\partial v}{\partial r} + \Delta\frac{\partial w}{\partial \zeta} = \text{tr}\left(\frac{\partial \mathcal{J}}{\partial t}\mathcal{J}^{-1}\right) = \frac{\partial}{\partial t}\log \det \mathcal{J}.$$

On the other hand we have

$$\Delta\left[\frac{2}{r}v\right] = \frac{2}{r + \eta^1}\frac{\partial \eta^1}{\partial t} = \frac{\partial}{\partial t}\log(r + \eta^1)^2.$$

Thus

$$\Delta\left[\frac{\partial v}{\partial r} + \frac{2}{r}v + \frac{\partial w}{\partial \zeta}\right] = \frac{\partial}{\partial t}\log(r + \eta^1)^2 \det \mathcal{J}. \quad (5.10)$$

and the equation (5.8) reads

$$\frac{\partial}{\partial t}\left[\log(\bar{\rho} + \Delta\rho)(r + \eta^1)^2 \det \mathcal{J}\right] = 0,$$

or, integrating this, we have

$$\bar{\rho}(r, \zeta) + \Delta\rho(t, r, \zeta) = (\bar{\rho}(r, \zeta) + \Delta\rho(t_0, r, \zeta))\left(\frac{r}{r + \eta^1(t, r, \zeta)}\right)^2 \det \mathcal{J}^{-1} \quad (5.11)$$

with

$$\det \mathcal{J} = \left(1 + \frac{\partial \eta^1}{\partial r}\right)\left(1 + \frac{\partial \eta^2}{\partial \zeta}\right) - \frac{\partial \eta^1}{\partial \zeta}\frac{\partial \eta^2}{\partial r}.$$

Now, on the region $\{\rho > 0\}$, the equations (5.3b)(5.3c) (5.3d) reduce to

$$\frac{Dv}{Dt} - \frac{r}{1 - \zeta^2}w^2 - r(1 - \zeta^2)(\Omega + \omega)^2 + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} = 0, \quad (5.12a)$$

$$\frac{Dw}{Dt} + \frac{2}{r}vw + \frac{\zeta}{1 - \zeta^2}w^2 + \zeta(1 - \zeta^2)(\Omega + \omega)^2 + \frac{1 - \zeta^2}{r^2}\left(\frac{1}{\rho}\frac{\partial P}{\partial \zeta} + \frac{\partial \Phi}{\partial \zeta}\right) = 0, \quad (5.12b)$$

$$\frac{D\omega}{Dt} + \frac{2}{r}v(\Omega + \omega) - \frac{2\zeta}{1 - \zeta^2}w(\Omega + \omega) = 0, \quad (5.12c)$$

Operating Δ on these equations, we get

$$\frac{\partial^2 \eta^1}{\partial t^2} - S_{11} - S_{12} + \Delta \left[\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} \right] = 0, \quad (5.13a)$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + 2S_{21} + S_{22} + S_{23} + \Delta \left[\frac{1 - \zeta^2}{r^2} \left(\frac{1}{\rho} \frac{\partial P}{\partial \zeta} + \frac{\partial \Phi}{\partial \zeta} \right) \right] = 0, \quad (5.13b)$$

$$\frac{\partial^2 \eta^3}{\partial t^2} + 2S_{31} - 2S_{32} = 0, \quad (5.13c)$$

where

$$S_{11} := \Delta \left[\frac{r}{1 - \zeta^2} w^2 \right] = \frac{r^*}{1 - \zeta^{*2}} \left(\frac{\partial \eta^2}{\partial t} \right)^2, \quad (5.14a)$$

$$\begin{aligned} S_{12} &:= \Delta [r(1 - \zeta^2)(\Omega + \omega)^2] = \\ &= ((1 - \zeta^2)\eta^1 - (r + \eta^1)(2\zeta + \eta^2)\eta^2)\Omega^2 + r^*(1 - \zeta^{*2}) \left(2\Omega + \frac{\partial \eta^3}{\partial t} \right) \frac{\partial \eta^3}{\partial t}, \end{aligned} \quad (5.14b)$$

$$S_{21} := \Delta \left[\frac{1}{r} v w \right] = \frac{1}{r^*} \frac{\partial \eta^1}{\partial t} \frac{\partial \eta^2}{\partial t}, \quad (5.14c)$$

$$S_{22} := \Delta \left[\frac{\zeta}{1 - \zeta^2} w^2 \right] = \frac{\zeta^*}{1 - \zeta^{*2}} \left(\frac{\partial \eta^2}{\partial t} \right)^2, \quad (5.14d)$$

$$\begin{aligned} S_{23} &:= \Delta [\zeta(1 - \zeta^2)(\Omega + \omega)^2] = \\ &= (1 - 3\zeta^2 - 3\zeta\eta^2 - (\eta^2)^2)\eta^2\Omega^2 + \zeta^*(1 - \zeta^{*2}) \left(2\Omega + \frac{\partial \eta^3}{\partial t} \right) \frac{\partial \eta^3}{\partial t}, \end{aligned} \quad (5.14e)$$

$$S_{31} := \Delta \left[\frac{1}{r} v(\Omega + \omega) \right] = \frac{1}{r^*} \frac{\partial \eta^1}{\partial t} \left(\Omega + \frac{\partial \eta^3}{\partial t} \right), \quad (5.14f)$$

$$S_{32} := \Delta \left[\frac{\zeta}{1 - \zeta^2} w(\Omega + \omega) \right] = \frac{\zeta^*}{1 - \zeta^{*2}} \frac{\partial \eta^2}{\partial t} \left(\Omega + \frac{\partial \eta^3}{\partial t} \right), \quad (5.14g)$$

where $r^* = r + \eta^1(t, r, \zeta)$, $\zeta^* = \zeta + \eta^2(t, r, \zeta)$.

The linearized approximation of the equations are

$$\frac{\partial}{\partial t} \left[\Delta \rho + \bar{\rho} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] \approx 0, \quad (5.15a)$$

$$\frac{\partial^2 \eta^1}{\partial t^2} - ((1 - \zeta^2)\eta^1 - 2r\zeta\eta^2)\Omega^2 - r(1 - \zeta^2) \cdot 2\Omega \frac{\partial \eta^3}{\partial t} + \Delta \left[\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} \right] \approx 0, \quad (5.15b)$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + (1 - 3\zeta^2)\eta^2\Omega^2 + \zeta(1 - \zeta^2) \cdot 2\Omega \frac{\partial \eta^3}{\partial t} + \Delta \left[\frac{1 - \zeta^2}{r^2} \left(\frac{1}{\rho} \frac{\partial P}{\partial \zeta} + \frac{\partial \Phi}{\partial \zeta} \right) \right] \approx 0, \quad (5.15c)$$

$$\frac{\partial^2 \eta^3}{\partial t^2} + \frac{2\Omega}{r} \frac{\partial \eta^1}{\partial t} - \frac{2\zeta\Omega}{1 - \zeta^2} \frac{\partial \eta^2}{\partial t} \approx 0. \quad (5.15d)$$

We have to analyze $\Delta\left[\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial\Phi}{\partial r}\right]$ and $\Delta\left[\frac{1-\zeta^2}{r^2}\left(\frac{1}{\rho}\frac{\partial P}{\partial\zeta} + \frac{\partial\Phi}{\partial\zeta}\right)\right]$.

To do it, assuming $P = A\rho^\gamma$, we use the variable $u = \frac{A\gamma}{\gamma-1}\rho^{\gamma-1}$ so that $dP/\rho = du$. Moreover we put

$$q(t, r, \zeta) := \frac{\Delta\rho(t, r, \zeta)}{\bar{\rho}(r, \zeta)} \quad (5.16)$$

$$\cong \mathring{q}(r, \zeta) - \left(\frac{\partial\eta^1}{\partial r} + \frac{2}{r}\eta^1 + \frac{\partial\eta^2}{\partial\zeta}\right) \quad (5.17)$$

so that

$$\Delta u = \bar{u}((1+q)^{\gamma-1} - 1). \quad (5.18)$$

Then we have

$$\begin{aligned} \Delta\left[\frac{1}{\rho}\frac{\partial P}{\partial r}\right] &= \Delta\frac{\partial u}{\partial r} = \\ &= \frac{\partial}{\partial r}(\Delta u) \cdot (\mathcal{J}^{-1})_1^1 + \frac{\partial}{\partial\zeta}(\Delta u) \cdot (\mathcal{J}^{-1})_1^2 + \frac{\partial\bar{u}}{\partial r}((\mathcal{J}^{-1})_1^1 - 1) + \frac{\partial\bar{u}}{\partial\zeta}(\mathcal{J}^{-1})_1^2, \end{aligned} \quad (5.19)$$

where \mathcal{J}^{-1} is the inverse of the matrix \mathcal{J} defined by (5.9). Thus

$$\Delta\left[\frac{1}{\rho}\frac{\partial P}{\partial r}\right] \cong (\gamma-1)\frac{\partial}{\partial r}(\bar{u}q) - \left(\frac{\partial\bar{u}}{\partial r}\frac{\partial\eta^1}{\partial r} + \frac{\partial\bar{u}}{\partial\zeta}\frac{\partial\eta^2}{\partial r}\right). \quad (5.20)$$

Let us define the operator \mathfrak{K} by

$$\mathfrak{K}f(r, \zeta) = \frac{1}{4\pi} \int_{-1}^1 \int_0^\infty K_{II}(r, \zeta, r', \zeta') f(r', \zeta') r'^2 dr' d\zeta' \quad (5.21)$$

so that

$$\Phi(t, r, \zeta) = -4\pi\mathfrak{G}\mathfrak{K}\rho(t, \cdot)(t, r, \zeta).$$

Moreover we define

$$\mathfrak{K}_r f(r, \zeta) = \frac{1}{4\pi} \int_{-1}^1 \int_0^\infty \frac{\partial}{\partial r} K_{II}(r, \zeta, r', \zeta') f(r', \zeta') r'^2 dr' d\zeta' \quad (5.22)$$

so that

$$\frac{\partial}{\partial r}\Phi(t, r, \zeta) = -4\pi\mathfrak{G}\mathfrak{K}_r\rho(t, \cdot)(t, r, \zeta).$$

Then, by the definition,

$$\Delta\frac{\partial\Phi}{\partial r} = -4\pi\mathfrak{G}(\mathfrak{K}_r\rho(t, \cdot)(r^*, \zeta^*) - \mathfrak{K}_r\bar{\rho}(r, \zeta)),$$

where $r^* = r + \eta^1(t, r, \zeta)$, $\zeta^* = \zeta + \eta^2(t, r, \zeta)$. But, applying (5.11), that is,

$$\rho(t, r'^*, \zeta'^*) = \overset{\circ}{\rho}(r', \zeta') \left(\frac{r'}{r'^*} \right)^2 \frac{1}{\det \mathcal{J}(t, r', \zeta')},$$

where $r'^* = r' + \eta^1(t, r', \zeta')$, $\zeta'^* = \zeta' + \eta^2(t, r', \zeta')$, to

$$\mathfrak{K}_{,r} \rho(t, \cdot)(r^*, \zeta^*) = \frac{1}{4\pi} \int_{-1}^1 \int_0^\infty \frac{\partial}{\partial r^*} K_{II}(r^*, \zeta^*, r'^*, \zeta'^*) \rho(t, r'^*, \zeta'^*) (r'^*)^2 dr'^* d\zeta'^*,$$

we get

$$\mathfrak{K}_{,r} \rho(t, \cdot)(r^*, \zeta^*) = \frac{1}{4\pi} \int_{-1}^1 \int_0^\infty \frac{\partial}{\partial r^*} K_{II}(r^*, \zeta^*, r'^*, \zeta'^*) \overset{\circ}{\rho}(r', \zeta') r'^2 dr' d\zeta'.$$

So, as for the linearized approximation, we see

$$\begin{aligned} \frac{\partial}{\partial r^*} K_{II}(r^*, \zeta^*, r'^*, \zeta'^*) &\cong \partial_r K_{II}(r, \zeta, r', \zeta') + \\ &+ \partial_r^2 K_{II}(r, \zeta, r', \zeta') \eta^1(t, r, \zeta) + \partial_\zeta \partial_r K_{II} \eta^2(t, r, \zeta) + \\ &+ \partial_{r'} \partial_r K_{II} \cdot (\eta^1)' + \partial_{\zeta'} \partial_r K_{II} \cdot (\eta^2)', \end{aligned}$$

where $(\eta^1)' = \eta^1(t, r', \zeta')$, $(\eta^2)' = \eta^2(t, r', \zeta')$. Then, by integration by parts, we get

$$\begin{aligned} &\int_{-1}^1 \int_0^\infty \partial_{r'} \partial_r K_{II} \cdot (\eta^1)' \overset{\circ}{\rho}(r', \zeta') r'^2 dr' d\zeta' + \int_{-1}^1 \int_0^\infty \partial_{\zeta'} \partial_r K_{II} \cdot (\eta^2)' \overset{\circ}{\rho}(r', \zeta') r'^2 dr' d\zeta' \\ &= - \int_{-1}^1 \int_0^\infty \partial_r K_{II}(r, \zeta, r', \zeta') \left[\frac{\partial}{\partial r'} (\overset{\circ}{\rho}(r', \zeta') (\eta^1)') + \frac{2}{r'} (\overset{\circ}{\rho})' (\eta^1)' + \frac{\partial}{\partial \zeta'} ((\overset{\circ}{\rho})' (\eta^2)') \right] r'^2 dr' d\zeta', \end{aligned}$$

using $\eta^2(t, r, \pm 1) = 0$. Thus we have

$$\begin{aligned} \Delta \frac{\partial \Phi}{\partial r} &\cong \frac{\partial \overset{\circ}{\Phi}}{\partial r} + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r^2} \eta^1 + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r \partial \zeta} \eta^2 + \\ &+ 4\pi \mathbf{G} \frac{\partial}{\partial r} \mathfrak{K} \left[\partial_r (\overset{\circ}{\rho} \eta^1) + \frac{2}{r} \overset{\circ}{\rho} \eta^1 + \partial_\zeta (\overset{\circ}{\rho} \eta^2) \right] - \frac{\partial \bar{\Phi}}{\partial r}. \end{aligned} \quad (5.23)$$

When we neglect $\overset{\circ}{q} = (\overset{\circ}{\rho} - \bar{\rho})/\bar{\rho}$, we can claim

$$\begin{aligned} \Delta \left[\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} \right] &\cong -(\gamma - 1) \frac{\partial}{\partial r} \left[\bar{u} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] - \left(\frac{\partial \bar{u}}{\partial r} \frac{\partial \eta^1}{\partial r} + \frac{\partial \bar{u}}{\partial \zeta} \frac{\partial \eta^2}{\partial r} \right) + \\ &+ \frac{\partial^2 \bar{\Phi}}{\partial r^2} \eta^1 + \frac{\partial^2 \bar{\Phi}}{\partial r \partial \zeta} \eta^2 + 4\pi \mathbf{G} \frac{\partial}{\partial r} \mathfrak{K} \left[\partial_r (\bar{\rho} \eta^1) + \frac{2}{r} \bar{\rho} \eta^1 + \partial_\zeta (\bar{\rho} \eta^2) \right]. \end{aligned} \quad (5.24)$$

On the other hand, we have

$$\begin{aligned}\Delta\left[\frac{1-\zeta^2}{r^2}\frac{1}{\rho}\frac{\partial P}{\partial\zeta}\right] &= \Delta\left[\frac{1-\zeta^2}{r^2}\frac{\partial u}{\partial\zeta}\right] = \\ &= \Delta\left[\frac{1-\zeta^2}{r^2}\right] \cdot \frac{\partial \bar{u}}{\partial\zeta} + \frac{1-\zeta^{*2}}{r^{*2}} \cdot \Delta\frac{\partial u}{\partial\zeta}.\end{aligned}\quad (5.25)$$

Here

$$\Delta\left[\frac{1-\zeta^2}{r^2}\right] = -\frac{2r(1-\zeta^2)\eta^1 + 2r^2\zeta\eta^2 + (1-\zeta^2)(\eta^1)^2 + r^2(\eta^2)^2}{r^2(r+\eta^1)^2} \quad (5.26)$$

$$\cong -\frac{2(1-\zeta^2)}{r^3}\eta^1 - \frac{2\zeta}{r^2}\eta^2, \quad (5.27)$$

and

$$\Delta\frac{\partial u}{\partial\zeta} = \frac{\partial}{\partial r}(\Delta u)(J^{-1})_2^1 + \frac{\partial}{\partial\zeta}(\Delta u)(J^{-1})_2^2 + \frac{\partial \bar{u}}{\partial r}(J^{-1})_2^1 + \frac{\partial \bar{u}}{\partial\zeta}((J^{-1})_2^2 - 1) \quad (5.28)$$

$$\cong (\gamma - 1)\frac{\partial}{\partial\zeta}(\bar{u}q) - \left(\frac{\partial \bar{u}}{\partial r}\frac{\partial \eta^1}{\partial\zeta} + \frac{\partial \bar{u}}{\partial\zeta}\frac{\partial \eta^2}{\partial\zeta}\right). \quad (5.29)$$

We have

$$\Delta\left[\frac{1-\zeta^2}{r^2}\frac{\partial \Phi}{\partial\zeta}\right] = \Delta\left[\frac{1-\zeta^2}{r^2}\right]\frac{\partial \bar{\Phi}}{\partial\zeta} + \frac{1-\zeta^{*2}}{r^{*2}}\Delta\frac{\partial \Phi}{\partial\zeta} \quad (5.30)$$

with

$$\begin{aligned}\Delta\frac{\partial \Phi}{\partial\zeta} &\cong \frac{\partial \bar{\Phi}}{\partial\zeta} + \frac{\partial^2 \bar{\Phi}}{\partial r \partial\zeta}\eta^1 + \frac{\partial^2 \bar{\Phi}}{\partial\zeta^2}\eta^2 + \\ &\quad + 4\pi\mathbf{G}\frac{\partial}{\partial\zeta}\mathfrak{K}\left[\partial_r(\bar{\rho}\eta^1) + \frac{2}{r}\bar{\rho}\eta^1 + \partial_\zeta(\bar{\rho}\eta^2)\right] - \frac{\partial \bar{\Phi}}{\partial\zeta}.\end{aligned}\quad (5.31)$$

When we neglect $\overset{\circ}{q} = (\overset{\circ}{\rho} - \bar{\rho})/\bar{\rho}$, we can claim

$$\begin{aligned}\frac{1}{\rho}\frac{\partial P}{\partial\zeta} + \frac{\partial \Phi}{\partial\zeta} &\cong -(\gamma - 1)\frac{\partial}{\partial\zeta}\left[\bar{u}\left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r}\eta^1 + \frac{\partial \eta^2}{\partial\zeta}\right)\right] - \left(\frac{\partial \bar{u}}{\partial r}\frac{\partial \eta^1}{\partial\zeta} + \frac{\partial \bar{u}}{\partial\zeta}\frac{\partial \eta^2}{\partial\zeta}\right) \\ &\quad + \frac{\partial^2 \bar{\Phi}}{\partial r \partial\zeta}\eta^1 + \frac{\partial^2 \bar{\Phi}}{\partial\zeta^2}\eta^2 + 4\pi\mathbf{G}\frac{\partial}{\partial\zeta}\mathfrak{K}\left[\partial_r(\bar{\rho}\eta^1) + \frac{2}{r}\bar{\rho}\eta^1 + \partial_\zeta(\bar{\rho}\eta^2)\right].\end{aligned}\quad (5.32)$$

The analysis of $\Delta\left[\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r}\right]$ and $\Delta\left[\frac{1-\zeta^2}{r^2}\left(\frac{1}{\rho}\frac{\partial P}{\partial\zeta} + \frac{\partial \Phi}{\partial\zeta}\right)\right]$ is complete,

and the linearized approximation (5.15a)-(5.15d) reads

$$\frac{\partial}{\partial t} \left[\Delta \rho + \bar{\rho} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] \approx 0, \quad (5.33a)$$

$$\begin{aligned} & \frac{\partial^2 \eta^1}{\partial t^2} - ((1 - \zeta^2) \eta^1 - 2r \zeta \eta^2) \Omega^2 - r(1 - \zeta^2) \cdot 2\Omega \frac{\partial \eta^3}{\partial t} + \\ & + (\gamma - 1) \frac{\partial}{\partial r} (\bar{u} \overset{\circ}{q}) + \frac{\partial \overset{\circ}{\Phi}}{\partial r} - \frac{\partial \bar{\Phi}}{\partial r} \\ & - (\gamma - 1) \frac{\partial}{\partial r} \left[\bar{u} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] - \left(\frac{\partial \bar{u}}{\partial r} \frac{\partial \eta^1}{\partial r} + \frac{\partial \bar{u}}{\partial \zeta} \frac{\partial \eta^2}{\partial r} \right) + \\ & + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r^2} \eta^1 + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r \partial \zeta} \eta^2 + 4\pi \mathbf{G} \frac{\partial}{\partial r} \mathfrak{K} \left[\partial_r (\overset{\circ}{\rho} \eta^1) + \frac{2}{r} \overset{\circ}{\rho} \eta^1 + \partial_\zeta (\overset{\circ}{\rho} \eta^2) \right] \approx 0, \end{aligned} \quad (5.33b)$$

$$\begin{aligned} & \frac{\partial^2 \eta^2}{\partial t^2} + (1 - 3\zeta^2) \eta^2 \Omega^2 + \zeta(1 - \zeta^2) \cdot 2\Omega \frac{\partial \eta^3}{\partial t} + \\ & - \left(\frac{2(1 - \zeta^2)}{r^3} \eta^1 + \frac{2\zeta}{r^2} \eta^2 \right) \left(\frac{\partial \bar{u}}{\partial \zeta} + \frac{\partial \bar{\Phi}}{\partial \zeta} \right) + \\ & + \frac{1 - \zeta^2}{r^2} \cdot \left\{ (\gamma - 1) \frac{\partial}{\partial \zeta} (\bar{u} \overset{\circ}{q}) + \frac{\partial \overset{\circ}{\Phi}}{\partial \zeta} - \frac{\partial \bar{\Phi}}{\partial \zeta} + \right. \\ & - (\gamma - 1) \frac{\partial}{\partial \zeta} \left[\bar{u} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] - \left(\frac{\partial \bar{u}}{\partial r} \frac{\partial \eta^1}{\partial \zeta} + \frac{\partial \bar{u}}{\partial \zeta} \frac{\partial \eta^2}{\partial \zeta} \right) \\ & \left. + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r \partial \zeta} \eta^1 + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial \zeta^2} \eta^2 + 4\pi \mathbf{G} \frac{\partial}{\partial \zeta} \mathfrak{K} \left[\partial_r (\overset{\circ}{\rho} \eta^1) + \frac{2}{r} \overset{\circ}{\rho} \eta^1 + \partial_\zeta (\overset{\circ}{\rho} \eta^2) \right] \right\} \approx 0, \end{aligned} \quad (5.33c)$$

$$\frac{\partial^2 \eta^3}{\partial t^2} + \frac{2\Omega}{r} \frac{\partial \eta^1}{\partial t} - \frac{2\zeta \Omega}{1 - \zeta^2} \frac{\partial \eta^2}{\partial t} \approx 0. \quad (5.33d)$$

If we use

$$\zeta(1 - \zeta^2) \Omega^2 + \frac{1 - \zeta^2}{r^2} \left(\frac{\partial \bar{u}}{\partial \zeta} + \frac{\partial \bar{\Phi}}{\partial \zeta} \right) = 0,$$

then (5.33c) can be written as

$$\begin{aligned} & \frac{\partial^2 \eta^2}{\partial t^2} + \frac{2\zeta(1 - \zeta^2)}{r} \Omega^2 \eta^1 + (1 - \zeta^2) \Omega^2 \eta^2 + \zeta(1 - \zeta^2) \cdot 2\Omega \frac{\partial \eta^3}{\partial t} + \\ & + \frac{1 - \zeta^2}{r^2} \cdot \left\{ (\gamma - 1) \frac{\partial}{\partial \zeta} (\bar{u} \overset{\circ}{q}) + \frac{\partial \overset{\circ}{\Phi}}{\partial \zeta} - \frac{\partial \bar{\Phi}}{\partial \zeta} + \right. \\ & - (\gamma - 1) \frac{\partial}{\partial \zeta} \left[\bar{u} \left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r} \eta^1 + \frac{\partial \eta^2}{\partial \zeta} \right) \right] - \left(\frac{\partial \bar{u}}{\partial r} \frac{\partial \eta^1}{\partial \zeta} + \frac{\partial \bar{u}}{\partial \zeta} \frac{\partial \eta^2}{\partial \zeta} \right) \\ & \left. + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial r \partial \zeta} \eta^1 + \frac{\partial^2 \overset{\circ}{\Phi}}{\partial \zeta^2} \eta^2 + 4\pi \mathbf{G} \frac{\partial}{\partial \zeta} \mathfrak{K} \left[\partial_r (\overset{\circ}{\rho} \eta^1) + \frac{2}{r} \overset{\circ}{\rho} \eta^1 + \partial_\zeta (\overset{\circ}{\rho} \eta^2) \right] \right\} \approx 0. \end{aligned} \quad (5.34)$$

Now the equation (5.33d) can be integrated immediately as

$$\frac{\partial \eta^3}{\partial t} = \mathring{\omega} - \frac{2\Omega}{r}\eta^1 + \frac{2\zeta\Omega}{1-\zeta^2}\eta^2, \quad (5.35)$$

where

$$\mathring{\omega} = \mathring{\omega}(r, \zeta) = \omega(t_0, r, \zeta) = \left. \frac{\partial \eta^3}{\partial t} \right|_{t=t_0} \quad (5.36)$$

is initial data. Here we note that $\eta^1 = \eta^2 = 0$ at $t = t_0$. Substituting (5.35) into (5.33b), (5.34), we get the final linearized system to be considered as

$$\begin{aligned} & \frac{\partial^2 \eta^1}{\partial t^2} - 2r(1-\zeta^2)\Omega \mathring{\omega} + 3(1-\zeta^2)\Omega^2 \eta^1 - 2r\zeta\Omega^2 \eta^2 + \\ & + (\gamma-1)\frac{\partial}{\partial r}(\bar{u}\mathring{q}) + \frac{\partial \mathring{\Phi}}{\partial r} - \frac{\partial \bar{\Phi}}{\partial r} + \\ & - (\gamma-1)\frac{\partial}{\partial r}\left[\bar{u}\left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r}\eta^1 + \frac{\partial \eta^2}{\partial \zeta}\right)\right] - \left(\frac{\partial \bar{u}}{\partial r}\frac{\partial \eta^1}{\partial r} + \frac{\partial \bar{u}}{\partial \zeta}\frac{\partial \eta^2}{\partial r}\right) + \\ & + \frac{\partial^2 \mathring{\Phi}}{\partial r^2}\eta^1 + \frac{\partial^2 \mathring{\Phi}}{\partial r \partial \zeta}\eta^2 + 4\pi\mathbf{G}\frac{\partial}{\partial r}\mathfrak{K}\left[\partial_r(\mathring{\rho}\eta^1) + \frac{2}{r}\mathring{\rho}\eta^1 + \partial_\zeta(\mathring{\rho}\eta^2)\right] = 0, \end{aligned} \quad (5.37a)$$

$$\begin{aligned} & \frac{\partial^2 \eta^2}{\partial t^2} + 2\zeta(1-\zeta^2)\Omega \mathring{\omega} - \frac{2\zeta(1-\zeta^2)}{r}\Omega^2 \eta^1 + (1+3\zeta^2)\Omega^2 \eta^2 + \\ & + \frac{1-\zeta^2}{r^2} \cdot \left\{ (\gamma-1)\frac{\partial}{\partial \zeta}(\bar{u}\mathring{q}) + \frac{\partial \mathring{\Phi}}{\partial \zeta} - \frac{\partial \bar{\Phi}}{\partial \zeta} + \right. \\ & - (\gamma-1)\frac{\partial}{\partial \zeta}\left[\bar{u}\left(\frac{\partial \eta^1}{\partial r} + \frac{2}{r}\eta^1 + \frac{\partial \eta^2}{\partial \zeta}\right)\right] - \left(\frac{\partial \bar{u}}{\partial r}\frac{\partial \eta^1}{\partial \zeta} + \frac{\partial \bar{u}}{\partial \zeta}\frac{\partial \eta^2}{\partial \zeta}\right) + \\ & \left. + \frac{\partial^2 \mathring{\Phi}}{\partial r \partial \zeta}\eta^1 + \frac{\partial^2 \mathring{\Phi}}{\partial \zeta^2}\eta^2 + 4\pi\mathbf{G}\frac{\partial}{\partial \zeta}\mathfrak{K}\left[\partial_r(\mathring{\rho}\eta^1) + \frac{2}{r}\mathring{\rho}\eta^1 + \partial_\zeta(\mathring{\rho}\eta^2)\right] \right\} = 0, \end{aligned} \quad (5.37b)$$

which is a closed system for unknown functions $\eta^1(t, r, \zeta)$ and $\eta^2(t, r, \zeta)$, provided that initial data $\mathring{q} = (\mathring{\rho} - \bar{\rho})/\bar{\rho}$ and $\mathring{\omega}$ are given.

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